QUIZ 8

(1) **(6 points)** Find the orders of the following elements in $\mathbb{Z}_{660}$ ($660 = 2^2 \cdot 3 \cdot 5 \cdot 11$).

(a) $7 \mod 660$
(b) $11 \mod 660$
(c) $35 \mod 660$
(d) $15 \mod 660$
(e) $34 \mod 660$
(f) $81 \mod 660$.

(2) **(3 points Bonus)** Prove that $n\mathbb{Z} \cap m\mathbb{Z} = \mathrm{lcm}(n,m)\mathbb{Z}$ where $\mathrm{lcm}(n,m)$ is the least common multiple of m and n (**hint** if you remember the proof given in class that $n\mathbb{Z} + m\mathbb{Z} = \gcd(n,m)\mathbb{Z}$ all you have to do is replace every instance of $n\mathbb{Z} + m\mathbb{Z}$ with $n\mathbb{Z} \cap m\mathbb{Z}$, $\subset$ with $\supset$, and $\gcd(n,m)$ with $\mathrm{lcm}(n,m)$).

(3) **(8 total points)** In class we briefly discussed how to make the cartesian product of two groups into a group. In detail if K and H are groups, then $G := K \times H$ becomes a group of order $|K||H|$ with multiplication defined as

$$(k_1, h_1) \cdot (k_2, h_2) := (k_1 k_2, h_1 h_2)$$

for all $k_1, k_2 \in K$ and $h_1, h_2 \in H$ where the products on the RHS are the products in K and H respectively. It is easy to show that this multiplication actually defines a group with 1_G equal to $(1_K, 1_H)$ and $(k,h)^{-1} = (k^{-1}, h^{-1})$. Moreover we have obvious (normal) subgroups of $K \times H$ which are isomorphic to K and H and have trivial intersection, namely

$$\{(k, 1_H) \in K \times H : k \in K\} \text{ and } \{(1_K, h) \in K \times H : h \in K\}$$

respectively.

(**WARNING** in the following problems be sure to use additive notation, so for instance the identity of $\mathbb{Z}_m \times \mathbb{Z}_n$ is $(0 \mod m, 0 \mod n)$ not $(1 \mod m, 1 \mod n)$)

(a) (**2 points**) Show that $\mathbb{Z}_3 \times \mathbb{Z}_2$ is cyclic by finding an explicit generator.
(b) (**2 point Bonus**) Show that $\mathbb{Z}_m \times \mathbb{Z}_n$ is always cyclic whenever m and n are relatively prime (**hint** Recall from a previous homework that under certain circumstances we have that $|ab| = \mathrm{lcm}(|a|, |b|)$ and note the orders of $(1 \mod m, 0 \mod n)$ and $(0 \mod m, 1 \mod n)$).
(c) (**2 points**) Show that $\mathbb{Z}_9 \times \mathbb{Z}_6$ is **not** cyclic.
(d) (**2 point Bonus**) Show that $\mathbb{Z}_m \times \mathbb{Z}_n$ is never cyclic whenever m and n are not relatively prime (**hint** Let d be a common divisor of m and n not equal to 1 and find two distinct (actually having trivial intersection) subgroups corresponding to d in $\mathbb{Z}_m \times \mathbb{Z}_n$ and state the relevant theorem which guarantees that $\mathbb{Z}_m \times \mathbb{Z}_n$ is not cyclic).