Here $q, q_1, q_2 \in \mathbb{H}$, $a, b, c, d, x, y, z, w \in \mathbb{R}$, $\overrightarrow{v} \in \mathbb{R}^3$ (identified with the last three components of a quaternion), $z_1, z_2 \in \mathbb{C}$.

(1) First we present three formulas which can be used to define quaternionic multiplication (depending on whether we think of $\mathbb{H}$ as $\mathbb{R}^4$, $\mathbb{C}^2$, or $\mathbb{R} \times \mathbb{R}^3$).

- $$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})\,(x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k})$$
$$= (ax - by - cz - dw) + (bx + ay + dz - cw)\mathbf{i} + (cx + az + bw - dy)\mathbf{j} + (aw + dx + cy - bz)\mathbf{k}$$
- $$(z_1, z_2)(w_1, w_2) = (z_1 w_1 - z_2 \overline{w}_2, z_1 w_2 + z_2 \overline{w_1})$$
- $$(a, \overrightarrow{v})(b, \overrightarrow{w}) = (ab - \overrightarrow{v} \cdot \overrightarrow{w}, a\overrightarrow{w} + b\overrightarrow{w} + \overrightarrow{v} \times \overrightarrow{w})$$

(2) Here are some formulas related to the conjugate.

- $$\overline{q_1 + q_2} = \overline{q}_1 + \overline{q}_2$$
- $$\overline{a} = a$$
- $$\overline{\overrightarrow{v}} = -\overrightarrow{v}$$
- $$\overline{aq} = a\overline{q}$$
- When $q \in \mathbb{S}^3$ (identified with quaternions whose length is 1) we have
$$\overline{q} = q^{-1}$$
- $$\overline{q}q = q\overline{q} = ||q||^2$$
- $$\overline{q_1 q_2} = \overline{q}_2\,\overline{q}_1 \quad \text{(notice the reverse!)}$$

(3) A few more general facts/ formulas

- When $q \in \mathbb{S}^2$ (identified with pure quaternions whose length is 1) we have
$$q^2 = -1$$
- $$(q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}$$