HOMEWORK 9

Throughout this homework all groups are finite.

(1) Suppose that H and K are subgroups of a group G.

(a) Show that the order of $H \cap K$ divides the order H and K.

Proof. $H \cap K < H$ and $H \cap K < K$. Thus, by Legrange's theorem, $H \cap K$ divides the order H and K. □

(b) Conclude that if the order of H and K are relatively prime, then $H \cap K = \{e_G\}$.

Proof. If two numbers are relatively prime, than the only number which divides them both is the number 1. But the only group of order 1 consists of the identity alone. □

(2) Suppose that $G = \langle x, y \rangle$ and that $yxy^{-1} = x^\beta$ for some integer β. Furthermore suppose that $|x| = a$ and $|y| = b$ (both are finite since G is finite). Lastly suppose that d is the first integer which satisfies $y^d \in \langle x \rangle$ (of course d could very well be equal to b).

(a) Show that β must be relatively prime to a and that $\beta^b \equiv 1 \mod a$. **hint** x^β must have order a and $y^i x y^{-i} = x^{\beta^i}$ as is easily checked by induction.

Proof. Since conjugation by y is an automorphism, the conjugate of any element of G by y has the same order as the original element. That x^β has order a. That is β has order a in $\mathbb{Z}_a$. But since the order of the element (any element) β in $\mathbb{Z}_a$ has order $a/gcd(a, \beta)$. Thus $gcd(a, \beta) = 1$, i.e. a and β are relatively prime.

We first show that $y^i x y^{-i} = x^{\beta^i}$ holds by induction. The $i = 1$ case holds by definition. Now suppose $y^n x y^{-n} = x^{\beta^i}$ holds. We now show the result holds for the case $n + 1$.

$$\begin{aligned}
y^{n+1} x y^{-(n+1)} &= (yy^n)\, x \left(y^{-n} y^{-1}\right) \\
&= y x^{\beta^n} y^{-1} && \text{(induction hyp)} \\
&= \left(yxy^{-1}\right)^{\beta^n} && \text{(conj is an iso)} \\
&= \left(x^\beta\right)^{\beta^n} \\
&= x^{\beta\beta^n} \\
&= x^{\beta^{n+1}}
\end{aligned}$$

In regards to the main question, since y has order b we must have that $y^b x y^b = exe^{-1} = x$. But from the equation just derived we also have $y^b x y^b = x^{\beta^b}$. Hence $e = x^{\beta^b - 1}$ and so a divides $\beta^b - 1$, i.e. $\beta^b \equiv 1 \mod a$.

□

(b) Show that $H := \langle x \rangle$ is a normal subgroup of G.

Proof. This was proven in more generality in the class on November 5^{th}. □

(c) Show that every left coset has the form $y^m H$ for some integer $0 \le m < d$ with no repeats.
hint Abstract on the following calculation

$$\begin{aligned}\left(x^3y^2xy^4\right)H &= \left(x^3H\right)\left(y^2H\right)(xH)\left(y^4H\right)\\ &= (eH)\left(y^2H\right)(eH)\left(y^4H\right)\\ &= \left(ey^2ey^4\right)H\\ &= y^6H.\end{aligned}$$

Proof. In class we showed that if $G = \langle x, y\rangle$, then $G/H = \langle xH, yH$. But since $x \in H$ we have that $xH = H = eH = e_{G/H}$. Thus $G/H = \langle yH$. Now I claim that the order of yH is d in G/H. Suppose that $(yH)^i = y^iH = H$. Then, $y^i \in H$. This means that y^i is a power of x and so $d \le i$. On the other hand, since $y^d \in H$, $y^dH = H$ and so the order of yH equals d. □

(d) Use LeGrange's theorem to conclude that $|G| = ad$.

Proof. H has order $|x| = a$ and G/H has order d. Thus by Legrange's theroem, $|G| = |H|\,|G/H| = ad$. □

(e) Explicitly use (2c) to show that every element of G can be written as $y^m x^n$ for integers $0 \le m < b$ and $0 \le n < a$.

Proof. Let $w \in G$. Then $w \in y^iH$ for some integer $0 \le m < d$. Hence $y^{-m}w \in H$ which means that $y^{-m}w$ is a power of x, say x^n. Therefore $w = y^m x^n$. □

(3) In this problem we show that G/H need not be isomorphic to a subgroup of G.

(a) Show that

$$Q_8/\langle -1\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Proof. The center of Q_8 is $\langle -1\rangle = 1, -1$ and so the subgroup in question is normal. Moreover

$$Q_8/\langle -1\rangle = \{\{1,-1\},\{i,-1\},\{j,-1\},\{k,-1\}\} = \{\langle -1\rangle, i\langle -1\rangle, j\langle -1\rangle, k\langle -1\rangle\}.$$

Moreover, since i, j and k squared all equal -1, we have that the square of every coset in the quotient is the identity. This means that the quotient is isomorphic to the stated group. □

(b) Show that no subgroup of Q_8 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (**hint** Q_8 has only one element of order 2).

Proof. $\mathbb{Z}_2 \times \mathbb{Z}_2$ has 3 elements of order 2 while Q_8 has only one. Therefore $\mathbb{Z}_2 \times \mathbb{Z}_2$ can not possibly be a subgroup of Q_8. □

(4) Let

$$G_1 := \langle (1,2,3,4), (1,3)(5,6,7,8) \rangle < S_8$$
$$G_2 := \langle e^{\imath\pi/4}, j \rangle < \mathbb{S}^3$$
$$G_3 := \langle (1,5)(3,7), (1,2,3,8)(4,5,6,7) \rangle < S_8$$
$$G_4 := \langle (1,2,3,4,5,6,7,8), (1,3)(2,6)(5,7) \rangle < S_8$$
$$G_5 := \langle (1,2,3,4,5,6,7,8), (1,5)(3,7) \rangle < S_8$$

In each of the 5 groups above we denote the first generator by x and the second by y.

(a) Show that each of the groups above has order 16. (**hint** Only G_3 does not fit into the pattern of problem 1 of this assignment. For this group instead note that $\langle x, yxy^{-1} \rangle$ is a normal subgroup of order 4 and index 4, and use LeGrange's theorem.)

Proof. (i) By direct calculation, in G_1, $|x| = |y| = 4$. Moreover, $yxy^{-1} = x^{-1} = x^3$. Thus we are in the situation of problem 2 with $a = 4, b = 4, \beta = 3$. Moreover since for any number i less than 4, y^i does not fix (say) the number 5, and every power of x does, we must have that $d = 4$. Hence, by problem 2 we have that $|G| = 4 \cdot 4 = 16$.

(ii) In G_2 we have $|x| = 8$ and $|y| = 4$. Moreover, $yxy^{-1} = x^{-1} = x^7$. Lastly, $j = y \notin \langle i \langle$ (since powers of x do not involve a j) but $j^2 = -1 = x^4$. Hence $d = 2$ and $|G| = 8 \cdot 2 = 16$.

(iii) In G_3 we have that $|x| = 2$ and $|y| = 4$. If we define z as yxy^{-1}, then another direct calculation shows that $\langle x, z \rangle$ is a group of order 4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ which we call H and that $yzy^{-1} = x$. Hence H is a normal subgroup. Lastly since y^i is never in H for $0 < i < 4$ we have that G/H has order 4. Hence, by Legrange's theorem, the order of $G_3 = 4 \cdot 4 = 16$.

(iv) (Moving faster) Here $a = 8$, $b = 2$, $\beta = 3$, and $d = 2$, so G_4 also has order 16.

(v) Lastly, in G_5 $a = 8$, $b = 2$, $\beta = 5$, and $d = 2$, so G_5 also has order 16.

□

(b) Show that

(i) $Z(G_1) = \langle x^2, y^2 \rangle = \{(1), x^2, y^2, x^2y^2\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $G_1/Z(G_1) = \langle xH, yH \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Since $yxy^{-1} = x^{-1}$ we have that $yx^2y^{-1} = x^{-2} = x^2$ we have that x^2 commutes with x (as always) and y and so x^2 is in the center. Next, not that $y^2xy^{-2} = yx^{-1}y^{-1} = \left(yxy^{-1}\right)^{-1} = \left(x^{-1}\right)^{-1} = x$. Hence y^2 commutes with itself, and as always y^2 commutes with y. Since the center is a group we also have that x^2y^2 is also in there. Finially, since G_1 is not abelian, we cannot have that $G_1/Z(G_1)$ is cyclic. Hence we have found all of the center (if the center were any bigger, the quotient would have order 2 or 1, and all groups of those orders are cyclic). Thus the quotient is a non-cyclic group of order 4. Since there is only one such group, the quotient is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

(ii) $Z(G_2) = \{1, -1\} \cong \mathbb{Z}_2$ and $G_2/Z(G_2) \cong D_4$.

Proof. One could do a direct calculation like the one done in the dihedral group homework assignment, but a more fun approach is as follows. We define $H := \{1, -1\}$. We want to show that H is the center of G_2. Clearly $H < Z(G_2)$, so H is normal and G_2/H is a group of order 8. Since

$$\begin{aligned} yxy^{-1}x^{-1} &= x^{-1}x^{-1} = x^{-2} = x^6 \notin H \\ &\Rightarrow yxy^{-1}x^{-1}H \neq H \\ &\Rightarrow (yH)(xH) \neq (xH)(yH) . \end{aligned}$$

so G/H is not abelian. Now there are only two non abelian groups of order 8, D_4 and Q_8. The latter has only 1 element of order 2. So if we show that the quotient has at least 2 elements of order 2, we must have the quotient is isomorphic to D_4. But $y^2 = j^2 = -1 \in H$, but $y \notin H$ implies that yH has order 2 in the quotient. Moreover, $(x^2)^2 = -1$ but $i = x^2 \notin H$. Finially, we need to verify that $yH \neq x^2H$. But this follows at once since $y^{-1}x^2 = -ji = k$ which does not equal 1 or -1. To make this isomorphism more explicit, we have $xH \mapsto a$ and $yH \mapsto b$ where a and b have their usual meanings as elements of D_4.
Lastly, if $z \in Z(G_2) - H$, then $zH \neq H$ and is in the center of G/H. As $G/H \cong D_4$, we must have that $zH = x^2H$. Hence $z \in \{x^2, -x^2\}$. But, $yx^2y^{-1} = x^6$ implies that neither possibilities for z are actually in the center and the proposed z cannot exist. $\square$

(iii) $Z(G_3) = \langle xyxy^{-1}, y^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Proof. In the notation above the two generators of the proposed center are xz and y^2. Since x commutes with z commute as well as itself, x commutes with xz. Moreover,

$$y(xz)y^{-1} = yxy^{-1}yz^{-1} = zx = xz$$

(where we have used the fact that x and z commute again). Hence $xz \in Z(G_3)$. Moreover $y^2xy^{-2} = yzy^{-1} = x$ implies that y^2 commutes with x, and since y^2 is a power of y it also commutes with y. Hence y^2 is also in the center. As in the first part, the center cannot be any larger, else the quotient of G_3 by its center would have order strictly less than 4, and all such groups are cyclic, we have found the entire center of G_3. $\square$

(iv) $Z(G_4) = \langle x^4 \rangle \cong \mathbb{Z}_2$.

Proof. We could do another "fun" approach, as the one used for G_2, but this time we do this the straight forward way. If $w = x^i y^j$ is in the center, then

$$\begin{aligned} w &= ywy^{-1} \\ &= yx^i y^{-1} y^j \\ &= \left(yxy^{-1}\right)^i y^j \\ &= \left(x^3\right)^i y^j \\ &= x^{3i} y^j \end{aligned}$$

which, after canceling the y's on the right, implies that $3i \equiv i \mod 8$ or $2i \equiv 0 \mod 8$. This forces i to equal either 0 or 4. We now do an independent calculation to restrict j. Since y has order 2 in G_4, there are only two possibilities for j, either $j = 0$ or $j = 1$. If $j = 1$ then,

$$\begin{aligned} w &= xwx^{-1} \\ &= x^i xyx^{-1} \\ &= x^i xyx^{-1} yy \\ &= x^i x \left(yxy\right)^{-1} y \\ &= x^i xxy \\ &= x^{i+2} y. \end{aligned}$$

Hence $x^i y = x^{i+2} y$ and again canceling the y's on the right, and then canceling the x^i on the left, shows us that $x^2 = e$, which is false. Hence the exponent of y in w must equal 0. Pairing this with our earlier calculation which restricted i to either 0 or 4 tells us that $w = e$ or $w = x^4$ as desired. □

(v) $Z(G_5) = \langle x^2 \rangle \cong \mathbb{Z}_4$ and $G_5/Z(G_5) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Since $yx^2y^{-1} = \left(yxy^{-1}\right)^2 = x^{5 \cdot 2} = x^1 0 = x^2$ we have that y commutes with x^2. Moreover, since x^2 is a power of x, x also commutes with y. Hence $\langle x^2 \rangle \subset Z(G_5)$. Since $\langle x^2 \rangle$ has order 4, this must be the entire center of G_5, else the quotient would again be cyclic. □

Show that

(i) $\langle x \rangle$ is normal in G_1 and calculate the quotient (i.e. tell me which familiar group the quotient is isomorphic to).

Proof. $yxy^{-1} = x^{-1}$, so the group generated by x is normal. The quotient is isomorphic to $\mathbb{Z}_4$ with generator $y\langle x \rangle$. □

(ii) $\langle x^2 \rangle$ is normal in G_2 and calculate quotient.

Proof. We have that $yx^2y^{-1} = x^{-2} = x^6 = \left(x^2\right)^3$. Thus the group H, generated by x^2 is a normal subgroup of order 4 (the order of $x^2 \in G_2$). Moreover, since x^2 (by definition) and y^2

(since $y^2 = -1 = x^4 = (x^2)^2$) are in $\langle x^2 \rangle$, but neither x nor y are, the quotient group

$$G_2/H = \langle xH, yH \rangle$$

is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (this is in the only group of order order 4 generated by elements of order 2). □

(iii) $\langle y^2 \rangle$ is normal in G_3 and calculate quotient.

Proof. $H := \langle y^2 \rangle$ is normal since it is a subgroup of the center. The quotient is a group of order 8. Since $xyx^{-1}y^{-1} = xz \notin \langle y^2 \rangle$, we have that the quotient is a non abelian group of order 8. Finally, since xH and yH are distinct elements of the quotient of order 2 (xH has order 2 since x does, and yH has order two since $y^2 \in H$, the quotient, G/H must be isomorphic to D_4. □

(iv) $\langle x^2 \rangle$ is normal in G_4 and calculate quotient.

Proof. Since $yx^2y^{-1} = \left(x^2\right)^3 = x^6 = \left(x^2\right)^3$, we have that $H := \langle x^2 \rangle$ is a normal subgroup of G_4. As x^2 has order 4, the quotient has order $16/4 = 4$ as well. Lastly, since xH and yH make up distinct elements of order 2, this quotient must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

(v) $\langle x^4 \rangle$ is normal in G_5 and calculate quotient.

Proof. The group $H := \langle x^4 \rangle$ is contained in the center, so it is normal. Since x^4 has order 2, H also has order 2 and the quotient therefore has order 8. Moreover, since

$$G/H = \langle xH, yH \rangle$$

and

$$\begin{aligned}(yH)(xH)(yH)^{-1}(xH)^{-1} &= \left(yxy^{-1}x^{-1}\right)H \\ &= x^5x^{-1}H \\ &= x^4H \\ &= eH \\ &\Rightarrow (xH)(yH) = (yH)(xH),\end{aligned}$$

the quotient must be abelian. Lastly, it is easy to see that

$$\left(x^iH\right)\left(y^jH\right) \mapsto (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_2$$

gives us an isomorphism between the quotient and $\mathbb{Z}_4 \times \mathbb{Z}_2$. □