

## HOMEWORK 9

Throughout this homework all groups are finite.

- (1) Suppose that  $H$  and  $K$  are subgroups of a group  $G$ .
  - (a) Show that the order of  $H \cap K$  divides the order  $H$  and  $K$ .
  - (b) Conclude that if the order of  $H$  and  $K$  are relatively prime, then  $H \cap K = \{e_G\}$ .
- (2) Suppose that  $G = \langle x, y \rangle$  and that  $yx y^{-1} = x^\beta$  for some integer  $\beta$ . Furthermore suppose that  $|x| = a$  and  $|y| = b$  (both are finite since  $G$  is finite). Lastly suppose that  $d$  is the first integer which satisfies  $y^d \in \langle x \rangle$  (of course  $d$  could very well be equal to  $b$ ).
  - (a) Show that  $\beta$  must be relatively prime to  $a$  and that  $\beta^b \equiv 1 \pmod{a}$ .  
**hint**  $x^\beta$  must have order  $a$  and  $y^i x y^{-i} = x^{\beta^i}$  as is easily checked by induction.
  - (b) Show that  $H := \langle x \rangle$  is a normal subgroup of  $G$ .
  - (c) Show that every left coset has the form  $y^m H$  for some integer  $0 \leq m < d$  with no repeats.  
**hint** Abstract on the following calculation

$$\begin{aligned}
 (x^3 y^2 x y^4) H &= (x^3 H) (y^2 H) (x H) (y^4 H) \\
 &= (e H) (y^2 H) (e H) (y^4 H) \\
 &= (e y^2 e y^4) H \\
 &= y^6 H.
 \end{aligned}$$

- (d) Use LeGrange's theorem to conclude that  $|G| = ad$ .
- (e) Explicitly use (2c) to show that every element of  $G$  can be written as  $y^m x^n$  for integers  $0 \leq m < b$  and  $0 \leq n < a$ .
- (f) Use  $Q_8 = \langle i, j \rangle$  to show that the form in part (2e) need not be unique.
- (g) If we have that

$$\langle x \rangle \cap \langle y \rangle = \{e_G\}$$

Show that the form in part (2e) is indeed unique.

- (3) In this problem we show that  $G/H$  need not be isomorphic to a subgroup of  $G$ .
  - (a) Show that

$$Q_8 / \langle -1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- (b) Show that no subgroup of  $Q_8$  is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  (**hint**  $Q_8$  has only one element of order 2).

(4) Let

$$G_1 := \langle (1, 2, 3, 4), (1, 3)(4, 5, 6, 7, 8) \rangle < S_8$$

$$G_2 := \langle e^{i\pi/4}, j \rangle < \mathbb{S}^3$$

$$G_3 := \langle (1, 5)(3, 7), (1, 2, 3, 8)(4, 5, 6, 7) \rangle < S_8$$

$$G_4 := \langle (1, 2, 3, 4, 5, 6, 7, 8), (1, 3)(2, 6)(5, 7) \rangle < S_8$$

$$G_5 := \langle (1, 2, 3, 4, 5, 6, 7, 8), (1, 5)(3, 7) \rangle < S_8$$

In each of the 5 groups above we denote the first generator by  $x$  and the second by  $y$ .

(a) Show that each of the groups above has order 16. (**hint** Only  $G_3$  does not fit into the pattern of problem 1 of this assignment. For this group instead note that  $\langle x, yxy^{-1} \rangle$  is a normal subgroup of order 4 and index 4, and use LeGrange's theorem.)

(b) Show that

$$(i) \ Z(G_1) = \langle x^2, y^2 \rangle = \{(1), x^2, y^2, x^2y^2\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ and } G_1/Z(G_1) = \langle xH, yH \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

$$(ii) \ Z(G_2) = \{1, -1\} \cong \mathbb{Z}_2 \text{ and } G_2/Z(G_2) \cong D_4 .$$

$$(iii) \ Z(G_3) = \langle xyxy^{-1}, y^2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$(iv) \ Z(G_4) = \langle x^4 \rangle \cong \mathbb{Z}_2.$$

$$(v) \ Z(G_5) = \langle x^2 \rangle \cong \mathbb{Z}_4 \text{ and } G_5/Z(G_5) \cong \mathbb{Z}_2 \times \mathbb{Z}_2. .$$

Show that

(i)  $\langle x \rangle$  is normal in  $G_1$  and calculate the quotient (i.e. tell me which familiar group the quotient is isomorphic to).

(ii)  $\langle x^2 \rangle$  is normal in  $G_2$  and calculate quotient.

(iii)  $\langle y^2 \rangle$  is normal in  $G_3$  and calculate quotient.

(iv)  $\langle x^2 \rangle$  is normal in  $G_4$  and calculate quotient.

(v)  $\langle x^4 \rangle$  is normal in  $G_5$  and calculate quotient.