HOMEWORK 9

As usual G and H are assumed to be groups.

(1) Let G_1, G_2 be groups.
   (a) Show that the map
   $$\begin{aligned}\pi_1 : G_1 \times G_2 &\to G_1 \\ (g_1, g_2) &\mapsto g_1\end{aligned}$$
   and
   $$\begin{aligned}\pi_2 : G_1 \times G_2 &\to G_2 \\ (g_1, g_2) &\mapsto g_2\end{aligned}$$
   are surjective homomorphisms.
   (b) Show that $\ker(\pi_1) \cong G_2$ and $\ker(\pi_2) \cong G_1$.
   (c) Conclude that $(G_1 \times G_2)/G_2 \cong G_1$ (really we are not modding out by G_1 here, but only a subgroup isomorphic to G_1).

(2) In this problem set we show that D_4 is not the direct product of two groups. Here we let H be an arbitrary proper (i.e. $H \neq D_4$ and $H \neq \{1\}$), normal subgroup of D_4.
   (a) Show that D_4/H is abelian. (**hint** what is the order of such of group and what do we know about groups of this order)
   (b) Use a problem from homework 8 to show that $[D_4, D_4] \subset H$.
   (c) Show that any two proper, normal subgroups must contain the element a^2 in their intersection.
   (d) Conclude that D_4 is not a non-trivial direct product of two groups.

(3) For H a normal subgroup of G, show that $G \to G/H$ is a surjective homomorphism of groups (this is true almost by definition).

(4) Show that $D_6 \cong D_3 \times \mathbb{Z}_2$.

(5) Show that if H is a normal subgroup of G of order 2, then H is contained in the center of G.

(6) Let T be a subset of $\{1, \ldots, n\}$.
   (a) Show that the set of permutations $\sigma \in S_n$ which satisfy $\sigma(T) = T$ is a subgroup of S_n which we denote by S_T.
   (b) Let $T^c := \{1, \ldots, n\} \backslash T$ (the complelement of T). Show that $S_{T^c} = S_T$ (i.e. the two groups are equal, not just isomorphic).
   (c) Show that $S_T \cong S_m \times S_{n-m}$ where m is the number of elements in T. (**hint**, consider the subgroups of S_T defined as the set of permutations, σ, which satisfy $\sigma(i) = i$ for every $i \in T^c$.)

(7) Use the fundamental theorem of finitely generated abelian groups to show that every finitely generated subgroup of $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m$ for some integer m (one can moreover show that we get the finitely generated bit for free, and that $m \leq n$, but this is harder).

(8) Suppose that H_1 is a normal subgroup of G_1 and H_2 is a normal subgroup of G_2. Then show that $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$.

(9) Suppose that $\varphi_i : G_i \to H$ $(i = 1, 2)$ are homomorphisms of groups. For $\varphi := (\varphi_1, \varphi_2)$ Define

$$G_1 \times_\varphi G_2 := \{(g_1, g_2) \in G_1 \times G_2 : \varphi_1(g_1) = \varphi_2(g_2)\}.$$

(**In parts (c)-(e) we assume that φ_1 is surjective**)

(a) Show that $G_1 \times_\varphi G_2$ is a subgroup of $G_1 \times G_2$.
(b) Show that if $G_1 \times_\varphi G_2 = G_1 \times G_2$ if and only if both φ_i's are trivial (i.e. send every element to 1_H).
(c) Show that the map π_2 from the first question in this assignment restricts to a surjective homomorpism

$$\pi_2| := \pi_2 \,|_{G_1 \times_\varphi G_2} : G_1 \times_\varphi G_2 \longrightarrow G_2.$$

(d) Show that $\ker(\pi_2|) \cong \ker(\varphi_1)$
(e) Use Lagrange's theorem and the first homomorphism theorem to show that

$$|G_1 \times_\varphi G_2| = \frac{|G_1|\,|G_2|}{|H|}.$$

(f) Let

$$\varphi_1 := sgn : S_3 \longrightarrow \mathbb{Z}_2$$

and

$$\varphi_2 := \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$$
$$a \mod 4 \mapsto a \mod 2$$

and define $T := S_3 \times_\varphi \mathbb{Z}_4$. Show that T is a non abelian group of order 12 which is not isomorphic to D_6 or A_4. (**hint** to show it is not isomorphic to D_6 show that T has an element of order 4, but D_6 does not; to show it is not isomorphic to A_4 show that T has an index 2 subgroup and use a previous homework problem to conclude that A_4 does not).
(g) Show that T from the previous problem does not contain a subgroup isomorphic to S_3 (count the number of elements of order 2!) but it does have a quotient isomorphic to S_3.

(10) Show that

(11) Suppose that H and K are normal subgroup of G. Show that

$$HK/H \cong K/H \cap K.$$

(Hint, define a map from K to HK/H by sending $k \in K$ to $kH \in HK/H$. Then show its surjective and compute its kernel).

(12) As a corollary of the previous problem, show that for any two integers n and m, that $nm = lcm(n,m)gcd(n,m)$ (hint, take $H = n\mathbb{Z}$ and $K = m\mathbb{Z}$ and compute the orders of both sides of the isomorphism in the previous problem).