HOMEWORK 8

As usual G and H are assumed to be groups.

(1) Let $g_1, g_2 \in G$ and define $[g_1, g_2]$ as $g_1 g_2 g_1^{-1} g_2^{-1}$. This is known as the commutator of g_1 and g_2 in G.

(a) Show that g_1 and g_2 commute if and only if $[g_1, g_2] = 1_G$ (this was a quiz question).

Sol'n

$$\begin{aligned} gh = hg &\Leftrightarrow \\ ghg^{-1} = h &\Leftrightarrow \\ ghg^{-1}h^{-1} = 1_G & \end{aligned}$$

(b) Show that $[g_1, g_2]^{-1} = [g_2, g_1]$.

Sol'n It suffices to show that $[g_1, g_2]\,[g_2, g_1] = 1_G$ (why??). But this is a calculation

$$\begin{aligned} [g_1, g_2]\,[g_2, g_1] &= \left(g_1 g_2 g_1^{-1} g_2^{-1}\right)\left(g_2 g_1 g_2^{-1} g_1^{-1}\right) \\ &= g_1 g_2 g_1^{-1}\left(g_2^{-1} g_2\right) g_1 g_2^{-1} g_1^{-1} \\ &= g_1 g_2 \left(g_1^{-1} g_1\right) g_2^{-1} g_1^{-1} \\ &= g_1 \left(g_2 g_2^{-1}\right) g_1^{-1} \\ &= g_1 g_1^{-1} \\ &= 1_G. \end{aligned}$$

(2) Denote the subgroup of G generated by the set $\{[g_1, g_2] : g_1, g_2 \in G\}$ be denoted by $[G, G]$. This is known as the *commutator* of G.

(a) In this series of problems we show that the commutator is normal.

(i) Show that G is abelian if and only if $[G, G] = \{1_G\}$.

Sol'n Obvious.

(ii) Show that if $\varphi : G \to H$ is a group homomorphism, then $\varphi([G, G]) \subset [H, H]$.

Sol'n We have

$$\begin{aligned} f([g_1, g_2]) &= f(g_1 g_2 g_1^{-1} g_2^{-1}) \\ &= f(g_1) f(g_2) f(g_1)^{-1} f(g_2)^{-1} \\ &= [f(g_1), f(g_2)] \in [H, H]\,. \end{aligned}$$

(Why is this enough to guarantee $\varphi([G, G]) \subset [H, H]$?)

(iii) A subgroup H of a group G is called *characteristic* if for any isomorphism f from G to itself (called an *automorphism*) we have that $f(H) \subset H$. Show that characteristic subgroups are normal subgroups.

Sol'n This follows at once from the fact that conjugation by any arbitrary element $g \in G$ is an automorphism of G (Fill in the details! For instance what does this have to do with a subgroup being normal??).

(iv) Use parts (2(a)ii) and (2(a)iii) to conclude that $[G, G]$ is a normal subgroup of G.

Sol'n The result is a consequence of the fact that an automorphism is a special kind of homomorphism.

(b) In this next series of problems we examine the group $G/[G, G]$ which is denoted as G_{ab} and is called the *abelianization* of G.

(i) Show that G_{ab} is abelian (so we didn't pick a crappy name for it!).

Sol'n The proof in (2(b)iv) goes the other way (fill in details).

(ii) Compute $(D_4)_{ab}$, $(A_4)_{ab}$, and $(S_4)_{ab}$. (You may want to do the remaining problems first). (**Hint** for $(S_4)_{ab}$ first show that all commutators are even. Then show that you can get a single 3-cycle as a commutator. Then use the fact that all 3-cycles are conjugate in S_4 to any given 3-cycle and the fact that $[S_4, S_4]$ is normal to show that $[S_4, S_4] = A_4$.)

Sol'n

As a general rule of thumb for these kinds of problems, one should try to calculate a bunch of commutators, and then make a conjecture as to what the commutator subgroup ought to be. However, in practice, there are far too many calculations to make ($|G|^2$ of them in fact) to actually compute this in practice. For instance in the first example (a and b have their usual meaning),

$$\begin{aligned} [a, b] &= aba^{-1}b^{-1} = aa = a^2, \\ \left[a^3, ab\right] &= a^3aba^{-3}(ab)^{-1} = a^3ababa^3 = a^4baba^3 \\ &= baba^3 = a^3a^3 = a^6 = a^2, \\ \left[a, a^2b\right] &= a(a^2b)a^{-1}(a^2b)^{-1} = a^3ba^{-1}ba^2 = a^3aa^2 = a^2, \end{aligned}$$

and

$$\left[a^2, ab\right] = a^2aba^{-2}(ab)^{-1} = 1$$

will hopefully begin to convince the reader that the commutator of D_4 consists of $\{1, a^2\}$ (however we have not proven it yet since we still need to do $64 - 4 = 60$ more calculations).

- $(D_4)_{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We claim that $[D_4, D_4] = \{1, a^2\} := H$. Indeed we have shown that in a previous homework assignment that this is the center of D_4. Thus in particular this group is normal. Since its quotient has order 4, the group must be isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. We can show it must be the latter in one of two ways (1) Since we are modding out by the center and since D_4 is not abelian we must have that the quotient is not cyclic, so it must be isomorphic to the latter group. (2) We can check directly that two elements have order 2 in the quotient (which is not the case in $\mathbb{Z}_4$. For instance $(aH)(aH) = a^2H = H$ since $a^2 \in H$ and $a \notin H$ while at the same time $(bH)(bH) = H$ and $b \notin H$. Finially note that $aH \neq bH$ since $ba \notin H$.

  We still need to show that the equality $\{1, a^2\} = [D_4, D_4]$ holds (we have only established $[D_4, D_4] \subset \{1, a^2\}$). But since H has order 2, the only possibility if $[D_4, D_4] \neq H$ is that $[D_4, D_4]$ is trivial. But this would imply that D_4 is abelian, which it is not.

  For a slightly more concrete proof of the fact that $\{1, a^2\} = [D_4, D_4]$ we use the fact that every element of D_4 has the form $a^m b^\epsilon$ for integers $0 \leq m < 4$ and $0 \leq \epsilon < 2$. So if $a^m b^\epsilon$, $a^{m'} b^{\epsilon'} \in D_4$ then

$$\begin{aligned}\left[a^m b^\epsilon, a^{m'} b^{\epsilon'}\right] &= (a^{m'} b^{\epsilon'})(a^{m'} b^{\epsilon'})(a^m b^\epsilon)^{-1}(a^{m'} b^{\epsilon'})^{-1} \\ &= (a^{m'} b^{\epsilon'})(a^{m'} b^{\epsilon'})(b^\epsilon a^{-m})(b^{\epsilon'} a^{-m'})\end{aligned}$$

  and compute for the four cases of $\epsilon = \{0, 1\}$ and $\epsilon' = \{0, 1\}$.
- $(A_4)_{ab} \cong \mathbb{Z}_3$. The subgroup $V := \{\epsilon, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4 of index $12/4 = 3$. Thus the quotient must be $\cong \mathbb{Z}_3$ which is abelian (there is only one group of order 3 since 3 is prime). Hence we have shown that $[A_4, A_4] \subset V$ by problem 2(b)iv below.
  We now show that any two non-identity elements of V are conjugate in A_4 and conclude that any normal subgroup of A_4 (which $[A_4, A_4]$ is) must contain all of V or no-nontrivial elements of V (a normal subgroup, by definition, must contain every element which is conjugate to a given element in it. That is a normal subgroup must be the union of conjugacy classes).

$$\begin{aligned}(123)(12)(34)(123)^{-1} &= (123)(12)(123)^{-1}(123)(34)(123)^{-1} \\ &= (23)(14) \qquad \text{(HW 4 q 2(a))} \\ &= (14)(23)\end{aligned}$$

at the same time

$$\begin{aligned}(123)^2(12)(34)(123)^{-2} &= (123)(14)(23)(123)^{-1} \\ &= (24)(13)\end{aligned}$$

- The fact that $[S_4, S_4] = A_4$ is a special case of the bonus. We give a direct proof of this. First note that $[S_4, S_4] \subset A_4$ since we have that S_4/A_4 is abelian. On the other hand $(12)(23)(12)^{-1}(23)^{-1} = (12)(23)(12)(23) = (123)^2 = (132)$ and $(23)(34)(23)^{-1}(34)^{-1} = (243)$. Hence $\langle (132), (243) \rangle \subset [S_4, S_4] \subset A_4$ (since the first group is generated by 2 commutators and the second group is the group generated by all commutators). However since $\langle (132) \rangle \subset \langle (132), (243) \rangle$ we have that $\langle (132), (243) \rangle$ has order divisible by 3 by Lagrange's theorem. Moreover in the previous homework (homework 7) you had shown that there is no subgroup of order 6 in A_4. Thus $\langle (132), (243) \rangle$ has order 3 or order 12 (these are the only other divisors of 12 which are divisible by 3). It can not have order three since if it did it would have to equal $\langle (132) \rangle$. But this is false since (243) is not in that group (it is not a power of (132)), but it is in $\langle (132), (243) \rangle$ by definition.

(iii) **Bonus** Show that $[S_n, S_n] = A_n$.

Sol'n Since $S_n/A_n \cong \mathbb{Z}_2$ we have that $[S_n, S_n] \subset A_n$ by part (2(b)iv).
The reverse inclusion takes more work. Just like we showed that all 2-cycles are conjugate, we can show that all 3-cycles are conjugate (actually all n-cycles are conjugate). Thus if a normal subgroup of S_n contains a single 3-cycle, then it must contain them all. Moreover it is easy to check that (CHECK IT!) that the set of 3-cycles generate A_n. Finailly, since $[(12), (23)] = (12)(23)(12)(23) = (13)(23) = (132)$ is a 3-cycle we have the desired result (I left out a boatload of details here! Make sure you fill them in if you want full credit!).

(iv) Show that if H is a normal subgroup of G and G/H is abelian, then $[G, G] \subset H$.

Sol'n Suppose that G/H is abelian and let g_1H and g_2H be arbitrary elements of G/H. Then

$$\begin{aligned}H &= 1_G H \\ &= [g_1H, g_2H] && (G/H \text{ assumed abelian}) \\ &= (g_1H)(g_2H)\left(g_1^{-1}H\right)\left(g_2^{-1}H\right) \\ &= \left(g_1g_2g_1^{-1}g_2^{-1}\right)H \\ &= [g_1, g_2]\, H.\end{aligned}$$

Thus $[g_1, g_2] = 1_G \mod H$ and so $[g_1, g_2] \in H$. Thus $[G, G] \subset H$.

(v) Show (by example of course!) that there is no such thing as a largest abelian subgroup (part 2(b)iv shows there is a largest abelian quotient).

Sol'n In the group D_4, the groups $\langle a \rangle$ and $\langle a^2, b \rangle$ are distinct abelian groups of order 4, neither of which is contained in a larger abelian group inside of D_4.

(3) **True of False** (This means if true, prove it; if false give a counter example.) Suppose that H is a normal subgroup of G

(a) If G is abelian, then G/H and H are abelian. **T** (H is abelian since it is a subgroup of G and to see that G/H is abelian simply note that for arbitary gH and kH in G/H we have that $(gH)(kH) = gkH = kgH = (kH)(gH)$.

(b) If H and G/H are abelian, then G is abelian. **F** (consider A_3 inside of S_3)

(c) If G is cyclic, then G/H and H are cyclic. **T**

(d) If H and G/H are cyclic, then G is cyclic. **F** (consider (again) A_3 inside of S_3)

(e) If G is generated by elements of order 2, then H is generated by elements of order 2. **F** (consider A_3 inside of S_3 (YET again))

(f) If G is generated by elements of order 2, then G/H is generated by elements of order 2. **T** (check that if $\{x_1, \ldots, x_n\}$ generates G, then $\{x_1H, \ldots, x_nH\}$ generates G/H. Then check that all x_1H's are either trivial or have order 2).