HOMEWORK 8

As usual G and H are assumed to be groups.

(1) Show that $D_6 \cong= D_2 \times \mathbb{Z}_2$.
  (a) Show that g_1 and g_2 commute if and only if $[g_1, g_2] = 1_G$ (this was a quiz question).
  (b) Show that $[g_1, g_2]^{-1} = [g_2, g_1]$.

(2) Denote the subgroup of G generated by the set $\{[g_1, g_2] : g_1, g_2 \in G\}$ be denoted by $[G, G]$. This is known as the *commutator* of G.
  (a) In this series of problems we show that the commutator is normal.
    (i) Show that G is abelian if and only if $[G, G] = \{1_G\}$.
    (ii) Show that if $\varphi : G \to H$ is a group homomorphism, then $\varphi([G, G]) \subset [H, H]$.
    (iii) A subgroup H of a group G is called *characteristic* if for any isomorphism f from G to itself (called an *automorphism*) we have that $f(H) \subset H$. Show that characteristic subgroups are normal subgroups.
    (iv) Use parts (**??**) and (**??**) to conclude that $[G, G]$ is a normal subgroup of G.
  (b) In this next series of problems we examine the group $G/[G, G]$ which is denoted as G_{ab} and is called the *abelianization* of G.
    (i) Show that G_{ab} is abelian (so we didn't pick a crappy name for it!).
    (ii) Compute $(D_4)_{ab}$, $(A_4)_{ab}$, and $(S_4)_{ab}$. (You may want to do the remaining problems first). (**Hint** for $(S_4)_{ab}$ first show that all commutators are even. Then show that you can get a single 3-cycle as a commutator. Then use the fact that all 3-cycles are conjugate in S_4 to any given 3-cycle and the fact that $[S_4, S_4]$ is normal to show that $[S_4, S_4] = A_4$.)
    (iii) **Bonus** Show that $[S_n, S_n] = A_n$.
    (iv) Show that if H is a normal subgroup of G and G/H is abelian, then $[G, G] \subset H$.
    (v) Show (by example of course!) that there is no such thing as a largest abelian subgroup (part **??** shows there is a largest abelian quotient).

(3) **True of False** (This means if true, prove it; if false give a counter example.) Suppose that H is a normal subgroup of G
  (a) If G is abelian, then G/H and H are abelian.
  (b) If H and G/H are abelian, then G is abelian.
  (c) If G is cyclic, then G/H and H are cyclic.
  (d) If H and G/H are cyclic, then G is cyclic.
  (e) If G is generated by elements of order 2, then H is generated by elements of order 2.

(f) If G is generated by elements of order 2, then G/H is generated by elements of order 2.