

HOMEWORK 7

As usual G is a group and $H < G$.

- (1) On the set G define a relation $\text{dom } H$ via

$$(g_1) \text{ dom } H(g_2) \Leftrightarrow g_1 g_2^{-1} \in H$$

. (Just like for left cosets defined in class we write $g_1 = g_2 \text{ dom } H$ instead of the awkward notation above).

- (a) Show that $\text{dom } H$ is an equivalence relation. (An equivalence class under this equivalence relation is called a right coset).

Sol'n

- If $g \in G$, then $gg^{-1} = 1_G \in H \Rightarrow g = g \text{ dom } H$.
- If $g = h \text{ dom } H$, then $gh^{-1} \in H \Rightarrow hg^{-1} = (gh^{-1})^{-1} \in H \Rightarrow h = g \text{ dom } H$.
- If $g = h \text{ dom } H$, $h = w \text{ dom } H \Rightarrow gh^{-1} \in H$ and $h^{-1}w \in H \Rightarrow gw^{-1} = (gh^{-1})(hw^{-1}) \in H \Rightarrow g = w \text{ dom } H$

- (b) Show that $g = 1 \text{ dom } H$ if and only if $g \in H$ (thus the right coset containing 1 equals the right coset of g if and only if $g \in H$).

Sol'n

$$g = 1_G \text{ dom } H \text{ if and only if } g1_G^{-1} = g \in H.$$

- (c) Show that the right coset of g under $\text{dom } H$ is $\{hg : h \in H\} := Hg$.

Sol'n

We temporarily denote the right coset of g by $[g]$. Now

$$\begin{aligned} g' \in [g] &\Leftrightarrow g = g' \text{ dom } H && \text{Def of an equivalence class} \\ &\Leftrightarrow g'(g)^{-1} \in H && \text{Def of dom H} \\ &\Leftrightarrow \exists h \in H \text{ so that } g'g^{-1} = h \\ &\Leftrightarrow \exists h \in H \text{ so that } g' = hg && \text{Right mult both sides by } g \\ &\Leftrightarrow g' \in Hg. && \text{Def of } Hg \end{aligned}$$

- (d) Let $H := \{1, b\}$ and $G := D_4$.

- (i) Compute the set of right cosets of H in G .

Sol'n

$\{\{1, b\}, \{a, ba\}, \{a^2, ba^2\}, \{a^3, ba^3\}\}$ which equal (in the same order) H (which equals Hb), Ha (which equals Hba), Ha^2 (which equals Hba^3), and Ha^3 (which equals Hba^3).

- (ii) Are the set of right cosets of H in G equal to the set of left cosets of H in G ? **Sol'n**

No. For instance $\{a, ba\}$ is a right coset but $a \neq ba \pmod H$ since $a^{-1}ba \notin H$.

- (iii) Cut a square into the 8 pieces which correspond to the 8 elements of D_4 . Shade regions in the same color if and only if they are in the same right coset of H . **Sol'n**

- (e) Show that $aH = Ha$ if and only if $a, a^{-1} \in N_G(H)$ (see the definition of $N_G(H)$ in homework assignment number 3). (The book gives a poor proof of this. Your proof should focus on the original definitions of gH and Hg as equivalence classes and show that each is contained in the other).

Sol'n Our solution will show that $aH \subset Ha \Leftrightarrow a$ normalizes H and $Ha \subset aH \Leftrightarrow a^{-1}$ normalizes H . While the book's argument that $aH = aHa^{-1} = Haa^{-1} = H$ is correct, it is similar to allowing a calc 1 student to plug in infinity into an improper integral without explicitly taking the limit (for instance I hardly think that multiplying entire sets on both the left and the right by elements of G is something that you are familiar with).

\Rightarrow Suppose that $aH = Ha$ and let $h \in H$ be arbitrary. Since h is an arbitrary element of H , we need to show that $aha^{-1} \in H$ and $a^{-1}ha \in H$ to conclude that $a, a^{-1} \in N_G(H)$. Now since $a^{-1}ah = h \in H$ we have that $ah \in aH$ which implies $ah \in Ha \Leftrightarrow ah = a \text{ dom } H \Leftrightarrow aha^{-1} \in H$. To show that $a^{-1}ha \in H$ is parallel, but I show it since I'm a nice guy. Since $ha \in Ha = aH \Leftrightarrow ha = a \pmod H \Leftrightarrow a^{-1}ha \in H$.

\Leftarrow For the converse let $g \in aH \Leftrightarrow a^{-1}g \in H \Rightarrow ga^{-1} = a(a^{-1}g)a^{-1} \in H$ where the last implication holds since a normalizes H . But $ga^{-1} \in H$ means that $g = a \text{ dom } H \Rightarrow g \in Ha$. Thus $aH \subset Ha$. We now will use that a^{-1} normalizes H to show that $Ha \subset aH$. Let $g \in Ha \Rightarrow g \text{ dom } a \Rightarrow ga^{-1} \in H \Rightarrow a^{-1}g = a^{-1}(ga^{-1})a \in H \Rightarrow a = g \pmod H \Rightarrow g \in aH$

- (f) Show that the following are equivalent:

- H is normal
- $gH = Hg$ for every element $g \in G$
- $gH = Hg$ for all g running through any set of left coset representatives.
- $gH = Hg$ for all g running through any set of right coset representatives.

Sol'n

(1) \Leftrightarrow (2) holds since if H is normal if and only if every element of G is in the normalizer of H . (2) \Rightarrow (3) and (2) \Rightarrow (4) are obvious. We prove (3) \Rightarrow (2) and simply note that (4) \Rightarrow (2) is the same. Let $t \in G$ be

arbitrary. Then $tH = gH$ for some left coset representative. Hence $tH = Hg$ by assumption. We need to show that $Hg = Ht$. But since the Hg 's are equivalence classes under an equivalence relation, they partition G and it suffices to show that $t \in Hg$ (I live in the state of IL since I live in the same state as you and you live in the state of IL). But $t = t1_G \in tH = Hg$ gives us the desired result.

- (2) Show that any subgroup of index 2 is normal in G . (The cleanest proof uses (1f) from the above)).

Sol'n Let H be an index 2 subgroup of G and let $a \notin H$. Then G is the disjoint union of H and aH . Thus the complement of H in G is aH . Similarly G is the disjoint union of H and Ha and thus the complement of H in G is Ha . Hence we have that $Ha = aH$. This coupled with the obvious fact that $H = H$ allows us to apply the above to conclude that H is normal.

- (3) In these problems we assume that H has index 2 in G with a set of left cosets $\{H, tH\}$.

(a) Show that
$$\begin{array}{c|cc} \cdot & H & tH \\ \hline H & H & tH \\ tH & tH & H \end{array}$$

where for instance the tH, tH entry equaling H means that any element of tH times any other element of tH is an element of H (compare this with say the example where G is S_n and H is A_n . In which case the table would read that any odd permutation times any other odd permutation is an even permutation).

Sol'n

- 1

- 11 Any $h_1, h_2 \in H$ we always have $h_1h_2 \in H$
- 12 For any $k \in tH$ and $h \in H$ we need to show that $hk \in tH$ which just means that we have to show that it is not in H (since H and tH partition G). But if $hk = h' \in H$ then $k = h'^{-1}h \in H$ and hence $k \in H \cap tH = \emptyset$ which is a contradiction.

- 2

- 21 If $k \in tH$ and $h \in H$, then $k = th'$ where $h' \in H$. Then $kh = th'h \in tH$.
- 22 Suppose that $k, k' \in tH$. We wish to show that $kk' \in H$. This just means that we need to show that $kk' \notin tH$. Since $k \in tH$ there exists an $h \in H$ so that $k = th$. Then

$$\begin{aligned} kk' \in tH &\Rightarrow kk' = th' \text{ for some } h' \in H \\ &\Rightarrow thk' = th' \\ &\Rightarrow hk' = h' \\ &\Rightarrow k' = h^{-1}h' \\ &\Rightarrow k' \in H \cap tH \end{aligned}$$

which contradicts the fact that distinct cosets are disjoint.

- (b) Define a binary operator on the set $\{H, tH\}$ so that it becomes a group of order two and so that $a \mapsto aH$ a surjective homomorphism of groups (**hint** the multiplication table is staring at you).

Sol'n

Just define the binary operator as in the table. In symbols we are defining $gH \cdot g'H := gg'H$. The map given is a homomorphism (almost by definition!!) since it takes gg' to $gg'H$ which is equal to $gH \cdot g'H$. The map is also a surjection since for instance $1_G \mapsto H$ and $t \mapsto tH$.

- (4) (a) Let $a, b \in D_4$ have their usual meaning as a rotation by 90 degrees and a reflection across a diagonal. Consider the set of cosets of $H := \langle b \rangle$ in $G := D_4$ discussed in class (we showed that they are $\{H, aH, a^2H, a^3H\}$ where for instance $aH = \{a, ab\}$). Show that $aa \in a^2H$ but $(ab)(ab) \notin a^2H$. Conclude that the rule

$$\begin{aligned} G/H \times G/H &\rightarrow G/H \\ (g_1H, g_2H) &\mapsto g_1g_2H \end{aligned}$$

is not single valued and hence does not define a function.

Sol'n

This is a simple inspection. The elements a and ab are in the same left coset (the coset aH above since $(ab)^{-1}a = ba^{-1}a = b \in H$, but a squared and ab squared are in different cosets. Therefore, multiplication does not induce a function on $G/H \times G/H \rightarrow G/H$ since it does not take equal cosets to equal cosets.

- (b) Show that H is not normal in G .

Sol'n

$b \in H$ but $aba^{-1} = ba^2 \notin H$ or note that the set of left cosets is not equal to the set of right cosets.

- (5) Same set up as the previous problem but now with $H := \langle a^2 \rangle = \{1, a^2\}$.
 (a) Prove that for any element of $t \in g_1H$ and any element of $s \in g_2H$ that $ts \in g_1g_2H$. (This is precisely what failed in the previous problem) (You shouldn't work out the 64 cases unless you have no idea of a shortcut!) Thus the statement $(g_1H) \cdot (g_2H) := g_1g_2H$ makes sense.

Sol'n

We begin by inspecting the cosets.

$$\begin{aligned} H &= \{1, a^2\} = a^2H & aH &= \{a, a^3\} = a^3H \\ bH &= \{b, a^2b\} = a^2bH & a^3bH &= \{a^3b, ab\} = abH \end{aligned}$$

I first list a couple of examples for the lost reader. Let's show that the product of any element of aH and bH (in that order) is in baH . There are 4 combinations of such elements

$$\begin{aligned} (a, b) : ab \in aH = abH & & (a^3, b) & : a^3b \in a^3bH = abH \\ (a, ba^2) : aba^2 = aa^2b = a^3b \in a^3bH = abH & & (a^3, ba^2) & : a^3ba^2 = a^5b = ab \in abH \end{aligned}$$

In theory one could continue like this (and by time they finish they would be damn good at multiplying element in D_4) but an easier way to approach this is to note that a^2 commutes with every element of D_4 . So if $t \in g_1H$ and $s = g_2 \pmod H$, then there exists elements t, t' which commute with every element of D_4 and are in H , with $s = g_1t$ and $t = g_2t'$. Thus $st = g_1tg_2t' = g_1g_2tt' \in g_1g_2H$ since $(g_1g_2)^{-1}(g_1g_2tt') = tt' \in H$

- (b) Set up a table in the same way that was done in (3a). Does this table remind you of a familiar group?

Sol'n

\cdot	H	aH	bH	abH
H	H	aH	bH	abH
aH	aH	H	abH	bH
bH	bH	abH	H	aH
abH	abH	bH	aH	H

The only entries that do not follow directly from the multiplication in D_4 are the (2,4),(3,2),(3,4),(4,2), and (4,4) entries. And

$$\begin{aligned} \text{(i) } (2,4) \quad aHabH &= a^2H = H & \text{(iii) } (3,4) \quad bHabH &= babH = \\ &\text{since } a^2 \in H & a^3H &= aH \text{ since } a^{-1}a^3 = \\ & & a^2 &\in H \\ \text{(ii) } (3,2) \quad bHaH &= baH = & \text{(iv) } (4,2) \quad abHaH &= abaH = \\ abH &\text{since } (ab)^{-1}ba = & aa^{-1}bH &= bH \\ b^{-1}a^{-1}ba &= ba^{-1}ba = a^2 \in & \text{(v) } (4,4) \quad abHabH &= ababH = \\ H. & & aa^{-1}H &= H. \end{aligned}$$

This table is the multiplication table for the set of symmetries of the letter H, i.e. the set of symmetries of a rectangle which is not a square, i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- (c) Show that H is normal in D_4 .

Sol'n Again we note that every element of H commutes with every element of D_4 (i.e. H is contained in the center of D_4). Thus if $w \in D_4$ and $t \in H$, then $wtw^{-1} = ww^{-1}t = t \in H$. (More generally this shows that any subgroup of the center of a group G is normal in G).

- (6) In this series of problems we show that A_4 does not contain a subgroup of order 6 (thus the converse of LeGrange's Theorem does not hold in general

¹). In the problem we assume that such a subgroup exists and refer to it as H .

- (a) Show that A_4 has 8 3-cycles by simply writing them all down.
- (b) Show that there exists a 3-cycle, s , not in H .
- (c) Conclude that H and sH make up the distinct cosets of H in A_4 (and hence have trivial intersection).
- (d) Use problem 3 to conclude that $s^3 \notin H$.
- (e) Draw your contradiction from the fact that $s^3 = \epsilon$ for any 3-cycle.
- (f) Conclude that such a subgroup H does not exist.

Sol'n

I basically lead you guys to the solution in this problem. The only subtle point is (6d). Now problem 3 says that the product of any element not in H with any other element not in H is an element of H . Thus for our 3-cycle s not in H we have that $s^2 \in H$. Also by the table we know that the product of any element in H with any element not in H is not an element of H (think back to the concrete example of even permutations and odd permutations). Hence $s^3 = s \cdot s^2 \notin H$. But since s is a 3-cycle we know that s^3 is the identity of A_4 . Hence we have shown that the identity of A_4 is not in H which contradicts the fact that H is a subgroup of A_4 .

¹We will show that the converse does indeed hold for subgroups of a group of order a power of a prime