HOMEWORK 7

As usual G is a group and $H < G$.

(1) On the set G define a relation $\operatorname{dom} H$ via

$$(g_1) \operatorname{dom} H (g_2) \Leftrightarrow g_1 g_2^{-1} \in H.$$

(Just like for left cosets defined in class we write $g_1 = g_2 \operatorname{dom} H$ instead of the awkward notation above).

(a) Show that $\operatorname{dom} H$ is an equivalence relation. (An equivalence class under this equivalence relation is called a right coset).

(b) Show that $g = 1_G \operatorname{dom} H$ if and only if $g \in H$ (thus the right coset containing 1 equals the right coset containing g if and only if $g \in H$).

(c) Show that the right coset of g under $\operatorname{dom} H$ is $\{hg : h \in H\}$.

(d) Let $H := \{1, b\}$ and $G := D_4$.

(i) Compute the set of right cosets of H in G.

(ii) Are the set of right cosets of H in G equal to the set of left cosets of H in G?

(iii) Cut a square into the 8 pieces which correspond to the 8 elements of D_4. Shade regions in the same color if and only if they are in the same right coset of H.

(e) Show that $aH = Ha$ if and only if a and $a^{-1} \in \mathrm{N}_G(H)$ (see the definition of $\mathrm{N}_G(H)$ in homework assignment number 3). (The book gives a poor proof of this. Your proof should focus on the original definitions of gH and Hg as equivalence classes and show that each is contained in the other).

(f) Show that the following are equivalent:

- H is normal
- $gH = Hg$ for every element $g \in G$
- $gH = Hg$ for all g running through any set of left coset representatives.
- $gH = Hg$ for all g running through any set of right coset representatives.

(2) Show than any subgroup of index 2 is normal in G. (The cleanest proof uses (1f) from the above)).

(3) In these problems we assume that H has index 2 in G with a set of left cosets $\{H, tH\}$.

(a) Show that

·	H	tH
H	H	tH
tH	tH	H

where for instance the tH, tH entry equaling H means that any element of tH times any other element of tH is an element of H (compare this with say the example where G is S_n and H is A_n. In which case the table would read that any odd permutation times any other odd permutation is an even permutation).

(b) Define a binary operator on the set $\{H, tH\}$ so that it becomes a group of order two (**hint** the multiplication table is staring at you).

(4) (a) Let $a, b \in D_4$ have their usual meaning as a rotation by 90 degrees and a reflection accross a diagonal. Consider the set of cosets of $H := \langle b \rangle$ in $G := D_4$ discussed in class (we showed that they are $\{H, aH, a^2H, a^3H\}$ where for instance $aH = \{a, ab\}$). Show that $aa \in a^2H$ but $(ab)(ab) \notin a^2H$. Conclude that the rule

$$G/H \times G/H \to G/H$$
$$(g_1H, g_2H) \mapsto g_1g_1H$$

is not single valued and hence does not define a function.
Conclude that what we did in the previous problem will not work for H and G as defined here.

(b) Show that H is not normal in G.

(5) Same set up as the previous problem but now with $H := \langle a^2 \rangle = \{1, a^2\}$.

(a) Let g_1 and g_2 be arbitrary elements of D_4. Prove that for any element $t \in g_1H$ and any element $s \in g_2H$ that $ts \in g_1g_2H$. (This is precisely what failed in the previous problem) (You shouldn't work out the 64 cases unless you have idea of a shortcut!) Thus the statement $(g_1H) \cdot (g_2H) := g_1g_2H$ makes sense.

(b) Set up a table in the same way that was done in (3a). (Don't waste too much time doing this. Since you know that the answer we get is independent of the representative we choose, pick ones that are convenient) Is this table the multiplication table of some group?

(c) Show that H is normal in D_4.

(6) In this series of problems we show that A_4 does not contain a subgroup of order 6 (thus the converse of LeGrange's Theorem does not hold in general [1]). In the problem we assume that such a subgroup exists and refer to it as H.

(a) Show that A_4 has 8 3-cycles by simply writing them all down.

(b) Show that there exists a 3-cycle, s, not in H.

(c) Conclude that H and sH make up the distinct cosets of H in A_4 (and hence have trivial intersection).

(d) Use problem 3 to conclude that $s^3 \notin H$.

(e) Draw your contradiction from the fact that $s^3 = \epsilon$ for any 3-cycle.

(f) Conclude that such a subgroup H does not exist.

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[1] We will show that the converse does indeed hold for subgroups of a group of order a power of a prime