HOMEWORK 6

Throughout G and H are (arbitrary) groups.

(1) Here are some basic properties of the order of an element for you to prove.
  (a) Why can't an element of a group have order 0? (Look at the definition of the order ... there's a word beginning with the letter "p")
  (b) Show that an element of a group has order 1 if and only if it is the identity.
  (c) Let $g \in G$. Show that if $g^2 = g^3 = e$, then $g = e$. Generalize this to show that if $g^m = g^n = e$ where m and n are relatively prime (i.e. have a greatest common divisor equal to 1) then $g = e_G$ **Hint**: Use a theorem proven in class on 2-28.
  (d) Show that $|g^{-1}| = |g|$.
  (e) Show that $q \in \mathbb{H}^*$ has order 4 if and only $q \in \mathbb{S}^2$ (i.e. pure quaternions of length 1).

(2) Recall that the center of G, denoted $Z(G)$ is defined as

$$\{z \in G : gz = zg \quad \forall g \in G\}$$

(i.e. it is the set of elements of G which commute with everybody in G). In the last homework (whether you realized it or not) you proved this was a subgroup of G because you showed it was the kernel of a homomorphism.
  (a) Instead give a direct proof of this by using a standard subgroup test (showing closure, etc) to show it is a subgroup.
  (b) Since the kernel of any homomorphism is always normal we already know that $Z(G)$ is a *normal* subgroup. Generalize this by showing that any subgroup of $Z(G)$ is also a normal subgroup of G.
  (c) Show that $Z(G)$ is an abelian group.
  (d) Show that G is abelian if and only if $Z(G) = G$ (the previous bullet gives you half of this for free).
  (e) Cite a problem from the very first homework assignment which showed that $Z(\mathbb{H}^*)$ is the set of real quaternions.
  (f) Give two proofs of the following: if $G = \langle X \rangle$, then show that g commutes with every element of X implies $g \in Z(G)$ [1] (the converse if obvious since $X \subset G$) . To give the concrete proof follow the following outline:
    - Show that assumption guarantees that g commutes with the inverse of every element of X. (What you're really showing is that in general g commutes with $h \in G$ implies that g commutes with h^{-1}. Either cite this in your notes or prove it again).
    - Show that the assumption guarantees that g commutes with any power of any element of X (the 1 power is by assumption, the -1 power was the above bullet point).

[1] This says, among other things that G is abelian if and only if any two elements of X commute

- Let $w \in G$ be an arbitrary element (hence a word in X), Use the above bullet to show that g commutes with w by writing w out as $x_1^{n_1} \cdots x_m^{n_m}$ and "dragging" (commuting) g by each $x_i^{n_i}$ individually . Perhaps an induction will make this "look" a little cleaner.

For the abstract proof (an abstract proof should never use any (group theoretical) words!)

- Show that $g \in Z(G)$ if and only if $C_G(g) = G$ (the centralizer of $g \in G$ i.e. the set of guys in G who commute with g) (this is simply a rephrasing of definitions).
- Show that the assumption that every element of X commutes with g implies that $X \subset C_G(g)$ (again, this completely obvious).
- Either reprove or cite in your notes (the day it was proved) the fact that $C_G(g) < G$ (this is the same place where we proved the first bullet of the concrete proof above). (**Bonus**: Give a "fancy" proof of this by showing that $C_G(g)$ is the set of fixed points of c_g discussed in the previous homework assignment).
- Use the third property of a group generated by a set (**gen3**) and the above bullet to prove that
$$G = \langle X \rangle < C_G(g) < G$$
which implies (by basic set theory) $C_G(g) = G$.
- Conclude from the first bullet point of this sequence that $g \in Z(G)$.

(g) Show that if $G \cong H$, then $Z(G) \cong Z(H)$ (**hint** If φ is an isomorphism between G and H show that $g \in Z(G)$ if and only if $\varphi(g) \in Z(H)$) (thus the isomorphism type, hence the order of the center of any group is an isomorphism invariant).

(3) In this problem, we prove that for any $n > 2$, that $Z(S_n)$ is trivial (i.e. contains only the identity).

(a) For concreteness we first outline the proof for $n = 5$ [2] and from there it should be easy to generalize. Define the elements of S_5 by
$$\sigma_1 = (2\,3\,4\,5) \qquad \sigma_2 = (1\,3\,4\,5)$$
$$\sigma_3 = (1\,2\,4\,5) \qquad \sigma_4 = (1\,2\,3\,5)$$
$$\sigma_5 = (1\,2\,3\,4).$$
Show that an element τ commutes with σ_i implies that $\tau(i) = i$ (this says that the function (... yes, elements of S_n are functions) τ maps i to itself.)

(b) Show that if τ is in the center, then we have $\tau(i) = i$ all $i = \{1, 2, 3, 4, 5\}$.

(c) Conclude that $\tau = (1)$.

(d) Generalize this in the case for general $n > 2$.

(4) Consider the group
$$G_1 = \langle (3\,6), (2\,4\,6)(1\,3\,5) \rangle \; (< S_6)$$
(as usual we denote the first generator listed, $(3\,6)$ as x and the second generator listed $(2\,4\,6)(1\,3\,5)$ as y.

[2] It should be just as easy to abstract a proof from the case $n = 4$ but 4 is a small enough number that you could possibly "cheat" and prove it by brute force!

(a) Show that G_1 is not abelian.
(b) Show that $z := (1\,4)(2\,5)(3\,6) \in G_1$. My thought process [3] for this was as follows: (1) xy is a 6-cycle since (you could of course do this the long way by composing functions from right to left)

$$\begin{aligned} xy &= (3\,6)(2\,4\,6)(1\,3\,5) && \\ &= (3\,6)(6\,2\,4)(1\,3\,5) && \text{Recycling} \\ &= (3\,6\,2\,4)(1\,3\,5) && \text{Cut \& Paste} \\ &= (6\,2\,4\;3)(3\,5\,1) && \text{Recycling} \\ &= (6\,2\,4\,3\,5\,1) && \text{Cut \& Paste} \end{aligned}$$

(2) any 6-cycle cubed is a 2,2,2-cycle which is the cycle type of our target, z. So a good place to start is by looking at $(xy)^3$.
(c) Show that $z \in Z(G)$ (use the previous problem!) (**Computational hint** Instead of computing xz, zx , yz, and zy you will probably find it easier to use the wonderful formula proven on 2-14 and discussed further on 2-16 pertaining to how to conjugate elements of S_n to show that $xzx^{-1} = z$ and $yzy^{-1} = z$ rather easily).
(d) Use maple or gap to show that this group has order 24.
(e) We already know 3 other groups of order 24, namely $B(\Delta^3)$, S_4, and $\mathbb{Z}_{24}$. Show that G_1 is not isomorphic to any of these groups by using the following hints:
- $B(\Delta^3)$ (as do all groups of quaternions) have at most one element of order two in them (namely the element -1).
- From the previous problem we know that $Z(S_4)$ is trivial.
- $\mathbb{Z}_{24}$ is abelian.

(5) Recall that

$$G_2 := B(\Box^3) = \left\langle \frac{\sqrt{2}(1+i)}{2}, \frac{\sqrt{2}(1+j)}{2} \right\rangle .$$

(This group has order $2 \cdot 24 = 48$) As usual we denote the first generator equal by x and the second by y.
(a) Show that i and j are in G_2. Conclude that $Q_8 < G_2$.
(b) Show that both $\langle i \rangle$ and $\langle j \rangle$ are not normal in G_2.
(c) Show however that Q_8 is normal in G_2.

(6) Define

$$G_3 = \langle (1\,5)(3\,7), (1\,2\,3\,8)(4\,5\,6\,7) \rangle .$$

(We'll be able to show later that this group has order 16. You could use maple or gap to verify this.) As usual we denote the first generator equal by x and the second by y.
(a) Show that G_3 is not abelian.
(b) Show both $z := (2\,6)(4\,8)$ and $w := (1\,5)(2\,6)(3\,7)(4\,8)$ are in G_3.
(c) Show that both y^2 and w are in the center of G.
(d) Show that the group $\langle x, z \rangle$ is a normal subgroup of G_3.
(e) Recall that BD_4 and D_8 are groups of order 16. Is either isomorphic to G_3? Why or why not?

[3] You might be able to find a much easier way of doing this, experiment a bit! This was simply the first thing that came to my mind

(7) Suppose that a and b are two distinct, commuting elements of order 2 (in any group). Show that $\langle a, b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.