

HOMEWORK 5

Prove the following.

- (1) Research or ask about the Quaternion group of order 8. Show explicitly that it is not isomorphic to D_4 (hint, count the number of elements of order 2 in both groups).

sol'n D_4 has 5 elements of order two but Q_8 has only 1.

- (2) Explicitly express D_4 as a subgroup of S_4 by following what D_4 does to the vertices of a square.

sol'n

$$\begin{aligned}1 &= \epsilon \\a &= (1234) \\a^2 &= (13)(24) \\a^3 &= (1432) \\b &= (12)(34) \\ba &= (13) \\ba^2 &= (14)(23) \\ba^3 &= (24)\end{aligned}$$

I get these permutations by labeling the vertices of the square $\{1, 2, 3, 4\}$ and computing $w(v)$ where v is a vertex and w is an element of D_4 considered as a rigid motion (hence a function) of \mathbb{R}^2 taking the unit square to itself.

- (3) Look up what the center of a group is in your textbook. Show that D_n has a non-trivial center if and only if n is even. (Hint: it may be helpful to first prove that $2x = 0 \pmod n$ has a solution if and only if n is even.)

sol'n For this problem I give two solutions, first an algebraic proof, and second more intuitive proof. I expect that most students will give the first proof.

algebraic proof If n is even, then $n/2$ is an integer and $w := a^{n/2} \in D_n$ and $n/2 = -n/2 \pmod n$. Thus

$$\begin{aligned}bwb^{-1} &= ba^{n/2}b^{-1} \\&= ba^{n/2}b \\&= a^{-n/2} \\&= a^{n/2} \\&= w.\end{aligned}$$

Thus b commutes with w . Moreover a also commutes with w since w is a power of a . And since any $g \in D_n$ can be written as a products of powers of a 's and b 's, w commutes with any element of D_n and by definition w is in the center of D_n .

We prove the converse by contraposition. Let w be now defined as $a^m b^\epsilon \in D_n$ with $0 \leq m < n$ and $\epsilon \in \{0, 1\}$ (in class we showed that every element of D_n can be written in this way) and we assume that w lies in the center of D_n . Thus in particular w must commute with b . Thus $bw = wb$ or $bw b^{-1} = w$. But

$$\begin{aligned} bw b^{-1} &= ba^m b^\epsilon b^{-1} \\ &= ba^m b^{-1} b^\epsilon \\ &= (bab^{-1})^m b^\epsilon \\ &= (a^{-1})^m b^\epsilon \\ &= a^{-m} b^\epsilon. \end{aligned}$$

Comparing both expressions for w , we see that $a^m = a^{-m}$ or $a^{2m} = 1_{D_n}$. This implies that n (the order of a) divides $2m$. But since m is assumed to be less than n , $2m$ is less than $2n$ and so $2m = n$ which implies that n is even.

geometric proof Pick a basis for \mathbb{R}^2 which consists of vectors, b_i in the fixed point set of s_i for $i = 1, 2$ and both live in the first quadrant of the plane. If w is a non-trivial element of the center of D_n , w must fix these two fixed point sets (which is the line spanned by the b_i 's) (since $w(\text{Fix}(s_i)) = \text{Fix}(ws_i w^{-1}) = \text{Fix}(s_i)$ for $i = 1, 2$) and since w preserves the length of a vector, w must take b_1 and b_2 to plus or minus themselves. Since the angle between these two basis vectors is strictly less than 90° we must have either w is the identity or w must take both basis vectors to their negatives (any other combination will make an obtuse angle). Hence in this basis, since w is not the identity by assumption, w must equal negative the identity matrix (that is w acts by multiplication by -1). However in any basis negative the identity matrix is still negative the identity matrix. In particular the matrix representing w in the standard basis must have this form. However multiplication by -1 can not fix any vertex (since it only fixes the origin which is not a vertex of any regular n -gon centered at the origin) and has order 2, and, as such, pairs the vertices of this regular n -gon and hence n must be even.

- (4) Write a subgroup of $GL(2, \mathbb{R})$ which is isomorphic to D_4 (hint: write the matrix which corresponds to the linear maps of \mathbb{R}^2 preserving the square).

sol'n

$$\begin{array}{ll} 1 & = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & a & = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ a^2 & = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & a^3 & = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ b & = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & ba & = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ ba^2 & = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & ba^3 & = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{array}$$

We get these matrices by thinking of elements of D_4 as rigid motions, which are linear maps and hence have matrix representations. For instance a is rotation by -90 degrees and hence takes $(1, 0)$ to $(-1, 0)$ and $(0, 1)$ to $(1, 0)$ and as such has the stated matrix form.