

## HOMEWORK 5

- (1) Suppose that  $X_1$  and  $X_2$  have the same number of elements (that is there exists a bijection from  $X_1$  to  $X_2$ ). Show that  $\text{Bij}(X_1)$  and  $\text{Bij}(X_2)$  are isomorphic. (**hint** If  $F$  is a bijection from  $X_1$  to  $X_2$  we want to take the function which takes  $x_1$  to  $x'_1$  to the function which takes  $F(x_1)$  to  $F(x'_1)$ . How did we do this for symmetric groups?) (this should be intuitive ... the ways one can permute a set should not depend on what you call the elements of the set).
- (2) Here we examine some basic properties of group homomorphisms. Here  $G, G_1, G_2$  and  $G_3$  are all groups. Also we consider  $\cong$  as a relation on the set of all groups. More precisely: define a relation  $\cong$  on the set (actually the class) of all groups via  $G_1 \cong G_2$  if and only if  $G_1$  and  $G_2$  are isomorphic.
- (a) Show that the identity map  $\text{Id}_G : G \rightarrow G$  is a group homomorphism (this should seem too obvious to prove. However, you should still have a string of equalities which start with  $\text{Id}_G(g_1g_2)$  and end with  $\text{Id}_G(g_1)\text{Id}_G(g_2)$ ). Use this to prove that  $\cong$  is a reflexive relation (i.e. any group is isomorphic to itself).
- (b) If  $\varphi : G_1 \rightarrow G_2$  is an isomorphism of groups, show that  $\varphi^{-1}$  is as well. (Need to answer the questions: why is  $\varphi^{-1}$  a bijection? Why does it preserve operations?) Use this to show that  $\cong$  is symmetric.
- (c) Suppose that

$$\varphi_1 : G_1 \rightarrow G_2 \text{ and } \varphi_2 : G_2 \rightarrow G_3$$

are group homomorphisms (isomorphisms). Show that

$$\varphi_2 \circ \varphi_1 : G_1 \rightarrow G_3$$

is also a group homomorphism (isomorphism). Use this to show that  $\cong$  is transitive.

(d) Conclude that  $\cong$  is an equivalence relation.

- (3) For any group  $G$ , a homomorphism from  $G$  to itself is called an *endomorphism*. The set of all endomorphisms of  $G$  is denoted by  $\text{End}(G)$ .
- (a) Show that for any group  $G$ ,  $\text{End}(G)$  is a monoid under the operation of function composition. (don't forget closure!)
- (b) For an endomorphism  $\varphi : G \rightarrow G$  show that the set

$$\{g \in G : \varphi(g) = g\}$$

forms a subgroup of  $G$  called the *fixed point subgroup* of  $\varphi$ .

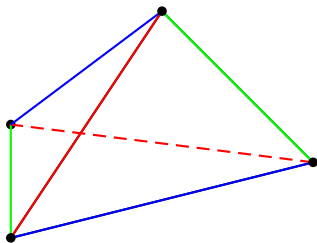
- (4) An invertible endomorphism (i.e. an isomorphism from a group to itself) is called an automorphism and the set of all automorphisms is denoted  $\text{Aut}(G)$ . This set is called the *automorphism group* of  $G$ .
- (a) Show that the automorphism group of  $G$  is (ironically!!) a group (since  $\text{Aut}(G) \subset \text{Bij}(G)$  you could use a subgroup test, or simply note that the automorphism group is the set of units in a monoid and state a theorem proven in class).

- (b) For any  $g \in G$ , define the function  $c_g : G \rightarrow G$  via  $k \mapsto gkg^{-1}$ . Show that  $c_g \in \text{Aut}(G)$  (i.e. show that  $c_g$  is an isomorphism).
- (c) Show that the assignment  $g \mapsto c_g$  is a homomorphism of groups  $G \rightarrow \text{Aut}(G)$ . (The image of this group is denoted  $\text{Inn}(G)$  and such automorphisms are called *inner*).
- (d) Show that the kernel of this homomorphism,  $g \mapsto c_g$ , is equal to the center of  $G$  (the center of  $G$ , denoted by  $Z(G)$ , is the intersection of all the centralizers of  $G$ , i.e. it is the set of elements of  $G$  which commute with every element of  $G$ ).
- (5) Suppose that  $G$  and  $H$  are groups. Show that  $G \times H \cong H \times G$ .
- (6) Recall from homework 3 problem (3) that the set of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with coefficients in any commutative ring with identity forms a group (this group is typically called the Heisenberg group). We denote this group  $UT(3, R)$ .

- (a) Show that all 26 non-identity elements of  $UT(3, \mathbb{Z}_3)$  have order 3 (just cube an arbitrary element of this group).
- (b) Show that all 26 non-identity elements of  $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$  also have order 3.
- (c) Show that despite this that  $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$  and  $UT(3, \mathbb{Z}_3)$  are not isomorphic.
- (d) True or False. If two groups,  $G$  and  $H$  have the same number of elements of any given order, then  $G$  and  $H$  are isomorphic.
- (7) Let  $S(\Delta^3)$  be the set of (orientation preserving) rigid motions of a (3-dimensional) tetrahedron. A tetrahedron has 6 edges. We can pair these edges (edges with the same color are paired)



- (a) Show that an element of  $S(\Delta^3)$  preserves this pairing (**hint** note that two edges are colored the same if and only if they do not touch).
- (b) If we label the green edges with a 1, red with a 2, and blue with a 3 we can have a homomorphism from  $S(\Delta^3)$  to  $S_3$ . What is the image of this homomorphism? What is the kernel? (For instance a right-handed rotation through line from the middle three vertices the upper vertex takes the pair of green edges to the set of blue edges, the set of blue edges to the set of red edges and the set of red edges back to the set of green edges and thus corresponds to the permutation (132).)