

HOMEWORK 4

To be safe, in this homework we always assume that n is a positive integer greater than 2.

- (1) Let G be a group and $g_1, g_2 \in G$. Define a relation \sim on G by declaring $g_1 \sim g_2$ if and only there exists a $t \in G$ so that $tg_1t^{-1} = g_2$. Show that \sim is an equivalence relation. Equivalence classes under this equivalence relation are known as *conjugacy classes* and two related elements are said to be *conjugate*.

Sol'n

Reflexive Let $g \in G$. Then $1_G \in G$ and $1_G g 1_G^{-1} = g$. Thus $g \sim g$ and \sim is reflexive.

Symmetric Let $g, h \in G$ and suppose that $g \sim h$. That is we are assuming that there exists a $t \in G$ so that $tgt^{-1} = h$. Then $t^{-1} \in G$ and

$$t^{-1}ht = t^{-1}(tgt^{-1})t = (t^{-1}t)g(t^{-1}t) = 1_G g 1_G = g$$

and $h \sim g$ and \sim is symmetric.

Transitive Suppose that $g, h, w \in G$ with $g \sim h$ and $h \sim w$. Then there exists a $t_1, t_2 \in G$ so that $t_1gt_1^{-1} = h$ and $t_2ht_2^{-1} = w$. Then, $t_2 \cdot t_1 \in G$ and $(t_2t_1)^{-1} = t_1^{-1}t_2^{-1}$. Thus

$$(t_2t_1)g(t_2t_1)^{-1} = t_2(t_1gt_1^{-1})t_2^{-1} = t_2ht_2^{-1} = w.$$

Thus $g \sim w$ and \sim is transitive.

- (2) (a) Let (ab) be a 2-cycle in S_n and τ be an arbitrary element of that same group. Show that $\tau(ab)\tau^{-1} = (\tau(a)\tau(b))$.

Sol'n We need to show two things, (1) that $x := \tau(ab)\tau^{-1}$ fixes all other numbers not equal to $\tau(a)$ or $\tau(b)$ and that $x(\tau(a)) = \tau(b)$. Suppose that $i \notin \{\tau(a), \tau(b)\}$. This means that (ab) fixes $\tau^{-1}(i)$, i.e. $(ab)\tau^{-1}(i) = \tau^{-1}(i)$ or $x(i) = \tau(ab)\tau^{-1}(i) = i$ and x fixes i as desired. Next we show that $x(\tau(a)) = \tau(b)$. But this is a straight forward calculation $x(\tau(a)) = \tau(ab)(a) = \tau(b)$.

- (b) Use (a) to show that any two 2-cycles are conjugate (it may be easier to show that any 2-cycle is conjugate to $(1\ 2)$ and then use problem (1) above), and conversely, if an element of S_n is conjugate to a 2-cycle, then it must itself be a 2-cycle (that is to say that the conjugacy class of a two cycle is the set of all 2-cycles).

Sol'n Let (ab) be a 2-cycle. Since conjugacy is an equivalence relation, we need only show that (ab) is conjugate to $(1\ 2)$. For this simply let $g = (1a)(2b)$ and use the previous part to conclude that $g(12)g^{-1} = (ab)$. Conversely, we use the previous part to conclude that if $w \in S_n$ is conjugate to $(1\ 2)$, then it must be a 2-cycle.

- (c) Show that if a normal subgroup contains a 2-cycle, then it must contain all 2-cycles.

Sol'n If N is a normal subgroup of S_n which contains a 2-cycle (ab) , then for any $g \in S_n$, $g(ab)g^{-1} \in N$ by the definition of a normal subgroup. But the previous part tells us that all two cycles have this form and as such N contains all 2-cycles.

- (d) Use the fact that 2-cycles generate S_n to conclude that if a normal subgroup contains a 2-cycle, then that group must be all of S_n .

Sol'n We have seen in class that the set of transpositions (and hence the set of all 2-cycles) generate S_n . By definition this means that if a subgroup contains all 2-cycles, then it must equal S_n . But in the previous part we have just shown that any normal subgroup which contains a 2-cycle has this property.

- (3) In this problem we show that S_n is generated by $b := (12 \cdots n)$ and $a := (12)$
 (a) Show that $b^m a b^{-m} = (m+1 \ m+2)$ (hint, figure where b^m sends 1 and 2 and then use (2a)) for all $m < n-2$.

Sol'n By induction it is clear that $b^m(1) = m+1$ and $b^m(2) = m+2$. Thus by (2a) we have that

$$b^m a b^{-m} = b^m(12)b^{-m} = (b(1) \ b(2)) = (m+1 \ m+2)$$

as desired.

- (b) Use part (a) to show that all such transpositions, $(i \ i+1)$, are in $\langle a, b \rangle$.

Sol'n Since $(i \ i+1) = b^{i-1} a b^{-(i-1)} \in \langle a, b \rangle$ (it's in $\langle a, b \rangle$ since every subgroup containing a and b must contain b^{i-1} , a and $b^{-(i-1)}$ and hence their product) we have the desired result that $(i \ i+1) \in \langle a, b \rangle$.

- (c) Use the fact that the set of transpositions generate S_n to conclude that $\langle a, b \rangle = S_n$.

Sol'n Let T be the set of all transpositions, then $T \subset \langle a, b \rangle$ by the above, and thus $S_n = \langle T \rangle \subset \langle a, b \rangle \subset S_n$ (the first equality holds since transpositions generate S_n , the first inclusion holds since a set is contained in a subgroup if and only if that group generated by that set is contained in the subgroup and the second inequality follows from the fact that all of this is happening in the larger group S_n) which implies that $\langle a, b \rangle = S_n$.

- (4) For the following problem state the order and parity (i.e. whether or not the element is even or odd) of each of the following elements. Also write each element as a product of transpositions.

(a) (124) (b) $(1456)(27)$ (c) $(123)(34)(45)$

(d) $(124)(35)$ (e) $(145)(24)$ (f) $(123)(34)$

- (5) Find 2 distinct, non-cyclic, proper subgroups of A_5 and show directly that they are not normal.