# HOMEWORK 4

Throughout this homework, $G$, $H$, and $K$ are groups. Also $\varphi$ is always a homomorphism of groups with domain $G$.

(1) In this problem you will fill in the details left out in class that the relation $G \cong H$ (i.e. $G$ and $H$ are isomorphic) is an equivalence relation on the collection of all groups.
   (a) Show that $Id_G : G \longrightarrow G$ is an isomorphism. Use this to show that $G \cong G$ and so $\cong$ is reflexive.
   (b) Show that if $\varphi : G \longrightarrow H$ is an isomorphism, then so is $\varphi^{-1} : H \longrightarrow G$. Use this to show that $\cong$ is symmetric.
   (c) Show that if $\varphi : G \longrightarrow H$ is an isomorphism and $\psi : H \longrightarrow K$ is an isomorphism, then $\psi \circ \varphi : G \longrightarrow K$ is an isomorphism. Use this to show that $\cong$ is transitive.

(2) For any two integers, show that $n\mathbb{Z}$ (set of multiples of $n$) and $m\mathbb{Z}$ are isomorphic (to make things look a bit neater, you only need to show that any of these groups are isomorphic to $\mathbb{Z}$, why??).

(3) Let
$$\ker(\varphi) := \{g \in G : \varphi(g) = e\}.$$
Show that $\ker(\varphi)$ is a group, called the kernel of $\varphi$.

(4) Define $\mathrm{PGL}(2, \mathbb{R})$ as the set of functions given by
$$\frac{ax + b}{cx + d}$$
with $a, b, c$ and $d$ real numbers such that $ad - bc \neq 0$. Consider the function given by
$$\pi : \mathrm{GL}(2, \mathbb{R}) \longrightarrow \mathrm{PGL}(2, \mathbb{R})$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ax + b}{cx + d}.$$
Show that $\pi$ is surjective and operation preserving (where the operation on the second set is taken as composition). Conclude from a theorem proven in class that $\mathrm{PGL}(2, \mathbb{R})$ forms a group under composition and $\pi$ is a homomorphism of groups. Find the kernel of $\pi$.

(5) Make tables which show the order of all 8 elements of $D_4$ and $Q_8$ (the quaternion group). Conclude that $D_4$ and $Q_8$ are not isomorphic (i.e. there is no isomorphism between these two groups).

(6) Show that $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$ by explicitly defining a map between the two (you do not need to show the map you define is an isomorphism however).

(7) Draw an object whose symmetry group is isomorphic to $\mathbb{Z}_4$.