HOMEWORK 4

To be safe, in this homework we always assume that n is a positive integer greater than 2.

(1) Let G be a group and $g_1, g_2 \in G$. Define a relation $\sim$ on G by declaring $g_1 \sim g_2$ if and only there exists a $t \in G$ so that $tg_1t^{-1} = g_2$. Show that $\sim$ is an equivalence relation. Equivalence classes under this equivalence relation are known as *conjugacy classes* and two related elements are said to be *conjugate*.

(2) (a) Let $(a\,b)$ be a 2-cycle in S_n and τ be an arbitrary element of that same group. Show that $\tau(a\,b)\tau^{-1} = (\tau(a)\,\tau(b))$.

(b) Use (a) to show that any two 2-cycles are conjugate (it may be easier to show that any 2-cycle is conjugate to (1 2) and then use problem (1) above), and conversely, if an element of S_n is conjugate to a 2-cycle, then it must itself be a 2-cycle (that is to say that the conjugacy class of a two cycle is the set of all 2-cycles).

(c) Show that if a normal subgroup contains a 2-cycle, then it must contain all 2-cycles.

(d) Use the fact that 2-cycles generate S_n to conclude that if a normal subgroup contains a 2-cycle, then that group must be all of S_n.

(3) In this problem we show that S_n is generated by $b := (12 \cdots n)$ and $a := (12)$

(a) Show that $b^m a b^{-m} = (m+1\ m+2)$ (hint, figure where b^m sends 1 and 2 and then use (2a)).

(b) Use part (a) to show that all such transpositions, $(i\,i{+}1)$, are in $\langle a, b\rangle$.

(c) Use the fact that the set of transpositions generate S_n to conclude that $\langle a, b\rangle = S_n$.

(4) For the following problem state the order and parity (i.e. whether or not the element is even or odd) of each of the following elements. Also write each element as a product of transpositions.

(a) (124)	(b) (1456)(27)	(c) (123)(34)(45)
(d) (124)(35)	(e) (145)(24)	(f) (123)(34)

(5) Find 2 distinct, non-cyclic, proper subgroups of A_5 and show directly that they are not normal.