

HOMEWORK 3 SOLUTIONS

- (1) The following exercise deals with the so called normalizer of a subgroup which is defined as $N_G(H) := \{g \in G \mid ghg^{-1} \in H \text{ for all } h \in H\}$ for a subgroup H of a finite group G .
- (a) Show $N_G(H)$ is also a subgroup of G (bonus, find an example where this does not hold if G is infinite).

Sol'n We use the test which is specific for finite groups. For this test we simply need to show that $N_G(H)$ is non-empty and is closed under multiplication. $N_G(H)$ is non-empty since for all $h \in H$, $1_G h 1_G^{-1} = h \in H$ which implies that $1_G \in N_G(H)$. Next we suppose that $g_1, g_2 \in N_G(H)$. Then for all $h \in H$ we have that

$$g_1 g_2 h g_2^{-1} g_1^{-1} = g_1 (h') g_1^{-1} \in H$$

where h' is some element of H since g_2 is in the normalizer of H .

Bonus Consider the subgroup of $GL(2, \mathbb{R})$ generated by

$$a := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } b := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We denote this subgroup as G .

Let

$$H := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

It is easy to see that $H = \langle b \rangle$ and hence $H < G$. I claim that $a \in N_G(H)$ but a^{-1} is not. This will imply that $N_G(H)$ is not a subgroup here (this does not contradict the above problem because H is infinite [it is clearly isomorphic to \mathbb{Z}]). Indeed

$$a^{-1} := \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \end{aligned}$$

but

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & n/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & n/2 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

The first calculation shows that $a \in N_G(H)$ (since an arbitrary element of H has the written form and the calculation shows that conjugating an arbitrary element of H by a produces a element of H) and if we take $n = 1$ in the second calculation, then that calculation shows that when conjugating

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(which is in H) by a^{-1} we get

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

which is not an element of H .

(b) Show that H is normal in $N_G(H)$.

Sol'n

We first need to show that $H < N_G(H)$. Since both H and $N_G(H)$ are groups this amounts to showing that $H \subset N_G(H)$. Let h_1 be an arbitrary element of H . We want to show that $h_1 \in N_G(H)$. This means we need to show that for an arbitrary $h \in H$ that $h_1 h h_1^{-1} \in H$. But since $h_1 \in H$ and H is a subgroup h_1^{-1} is also in H . Also since H is a subgroup it is closed under multiplication so $h_1 h h_1^{-1} \in H$ (it is the product of 3 elements of H). Thus $h_1 \in N_G(H)$ and since h_1 was an arbitrary element of H , $H \subset N_G(H)$.

The rest is almost true by definition. In detail; let $h \in H$ and $g \in N_G(H)$ be arbitrary. Then since $g \in N_G(H)$ we have that $ghg^{-1} \in H$. But this is exactly what it means for a subgroup H to be normal.

(c) Show that $N_G(H)$ is the largest subgroup of G with H normal in $N_G(H)$ in the sense that if K is a subgroup of G and H is normal in K , then $K \subseteq N_G(H)$.

Sol'n

Suppose that K is a subgroup of G which contains H as a normal subgroup. Then let $k \in K$. We want to show that $k \in N_G(H)$. To this end let $h \in H$ be arbitrary. Since H is normal in K we have that $khk^{-1} \in H$. This means that $k \in N_G(H)$ as desired.

- (d) Let $G = S_3$ and H be the subgroup generated by the bijection which exchanges 1 and 2. Find the normalizer of H in S_3 .

Sol'n

(At the time of this homework assignment we hadn't developed cycle notation yet, however we still use it here. The only difference lies in the amount of writing we would have to do.) We are considering the subgroup $H := \langle (12) \rangle < S_3$. Since $(12)(12) = \epsilon$ we have that (12) has order 2 and $\langle (12) \rangle = \{\epsilon, (12)\}$. Since if $g \in S_3$ satisfies $g(12)g^{-1}$ cannot equal the identity, ϵ ¹. we simply need to investigate which elements of S_3 satisfy $g(12)g^{-1} = (12)$. We only need to test the elements of $S_3 \setminus H$ since we already know H is contained in the normalizer.

$$(23)(12)(23)^{-1} = (13)$$

$$(13)(12)(13)^{-1} = (23)$$

$$(123)(12)(123)^{-1} = (23)$$

$$(132)(12)(132)^{-1} = (13)$$

and since none of these elements satisfy $g(12)g^{-1} = (12)$, the normalizer of H consists only of H itself.

- (2) In the following problem let H and N be subgroups of a group G .
 (a) Show that the intersection of N and H is a subgroup of G . Also show that it is the largest subgroup of G which is contained in both H and N .

Sol'n

- Since H and N are subgroups, $1_G \in N$ and H and hence $1_G \in N \cap H$. Thus $N \cap H$ is non-empty.
- Now if $g_1, g_2 \in N \cap H$, then g_1 and g_2 are in both N and H which means that g_1g_2 are in both N and H (since both are subgroups of G and are hence closed under multiplication). And hence $N \cap H$ is closed under multiplication.
- Let $g \in N \cap H$. Then $g \in N$ and $g \in H$. Since N is a subgroup $g^{-1} \in N$. Also since H is a subgroup we know that $g^{-1} \in H$. Hence $g^{-1} \in N \cap H$.

Therefore by the 2 step subgroup test we have that $H \cap N$ is a subgroup of G .

Next we show that it is the largest subgroup which is contained in both H and N . Indeed this is true of sets in general. In more detail $K \subset H$ and $K \subset N \Leftrightarrow K \subset H \cap N$.

¹in general $ghg^{-1} = 1_G \Rightarrow hg^{-1} = g^{-1} \Rightarrow h = hg^{-1}g = g^{-1}g = 1_G \Rightarrow h = 1_G$

(b) Suppose in addition that N is normal in G . Then show

$$HN = \{hn \mid h \in H, n \in N\}$$

is also a subgroup of G . Also show that it is the smallest subgroup of G which contains both H and N .

Sol'n

- Since H and N are subgroups, $1_G \in N$ and H and hence $1_G \in HN$. Thus $1_G = 1_G 1_G \in HN$ and HN is non-empty.
- Now if $g_1, g_2 \in HN$, then $g_1 = h_1 n_1$ where $h_1 \in H$ and $n_1 \in N$ and $g_2 = h_2 n_2$ where $h_2 \in H$ and $n_2 \in N$. Moreover since N is normal in G (actually you only need that H is contained in the normalizer of N in G) we have that

$$h_2^{-1} n_1 h_2 = n \text{ for some } n \in N.$$

Hence

$$\begin{aligned} g_1 g_2 &= (h_1 n_1)(h_2 n_2) \\ &= (h_1 1_G n_1 h_2 n_2) \\ &= (h_1 h_2 h_2^{-1} n_1 h_2 n_2) \\ &= h_1 h_2 (h_2^{-1} n_1 h_2) n_2 \\ &= h_1 h_2 n n_2 \\ &= h' n' \end{aligned}$$

where $h' = h_1 h_2 \in H$ and $n' = n n_2 \in N$. Hence $g_1 g_2 \in HN$ and HN is closed under multiplication.

- Let $g \in HN$. Then $g = hn$ where $n \in N$ and $h \in H$. Since N is normal in G , we have that $n^{-1} \in N$ and $hn^{-1}h^{-1} = n'$ for some $n' \in N$. Thus

$$\begin{aligned} g^{-1} &= (hn)^{-1} \\ &= n^{-1} h^{-1} \\ &= h^{-1} h n^{-1} h^{-1} \\ &= h^{-1} n'. \end{aligned}$$

And since H is a subgroup of G , $h^{-1} \in H$ and thus $g^{-1} \in HN$. Therefore by the 2 step subgroup test we have that HN is a subgroup of G .

Next we show that it is the smallest subgroup which contains both H and N . To this end suppose that K is a subgroup of G which contains both H and N . Then for any $g \in HN$ we have that $g = hn$ with $h \in H$ and $n \in N$. Thus $h \in K$ and $n \in K$. As K is a subgroup we have that $hk \in K$. But $hk = g$ and so $g \in K$. Since g was an arbitrary element of HN we have that $HN \subset K$.

(3) This problem deals with the subset

$$SO(2, \mathbb{R}) := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

of $M(2, \mathbb{R})$.

- (a) Show that $SO(2, \mathbb{R})$ is an abelian subgroup of $SL(2, \mathbb{R})$ where the operation is matrix multiplication (you might need to look up some trig identities).

Sol'n

For ease of writing we denote $SO(2, \mathbb{R})$ as simply SO . Define

$$A_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Then an easy calculation shows that $A_\theta A_\phi = A_{\theta+\phi}$ (this shows that the map $\mathbb{R} \rightarrow SO$ given by $\theta \mapsto A_\theta$ is a homomorphism). Since SO is subset of $SL(2, \mathbb{R}) (= SL)$ ($\sin^2 + \cos^2 = 1$ implies that all elements of SO have determinant 1) we can apply the one step subgroup test (since SO is clearly not empty).

- If X, Y are in SO , then $X = A_\theta$ and $Y = A_\phi$ for some real numbers θ and ϕ . Since $Y A_{-\phi} = A_{\phi-\phi} = A_0 = Id$ we have that $Y^{-1} = A_{-\phi}$. Hence $XY^{-1} = A_\theta A_{-\phi} = A_{\theta-\phi} \in SO$. Therefore, by the one step subgroup test SO is a subgroup of SL .
- (b) Is this subgroup isomorphic to \mathbb{Q}^* or \mathbb{R}^* under multiplication? Explain why or why not.

Sol'n No. This follows at once since neither of the groups listed have an element of order 4 but the group SO does.

- (c) Give a geometric interpretation of this group.

Sol'n You can think of this group as the set of all rotations in the plane which fix the origin. Alternatively we can think of this group as the unit circle.

- (d) Show that it is not a normal subgroup of $SL(2, \mathbb{R})$.

Sol'n We just need to find an element of $A \in SO$ and $B \in SL$ so that $BAB^{-1} \notin SO$. Indeed if we let

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$B := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

we have that then $A \in SO$ with $\theta = \pi/2$ and $B \in SL(2, \mathbb{R})$ (simply note that the determinant of B is 1). However

$$\begin{aligned}
BAB^{-1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}
\end{aligned}$$

Now this element can not possibly be in $SO(2, \mathbb{R})$ since all entries in any matrix in this group must have entries which lie between -1 and 1 .

- (4) Suppose that $g, h \in G$ are two commuting elements which generate subgroups of G which meet only at the identity.
- (a) Show that the order of gh is the least common multiple of the orders of g and h .

Sol'n Solution given in class.

- (b) Show that both hypothesis on g and h are necessary (part of your grade in this problem includes figuring out the logic of exactly what this means and how to prove without doing either too much work or too little work).