HOMEWORK 3

Throughout the assignment R denotes a commutative ring (with identity). Recall there is an abstract definition of this, but for our purposes you can think of R as being $\mathbb{Z}$, $\mathbb{Z}_n$ (for some integer n), $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

(1) Let $G := \mathbb{R} - \{-1\}$ and define $a * b := a + b + a \cdot b$ (here $+$ and $\cdot$ have their usual meanings). Show that $(G, *)$ is a group. (Students typically forget a lot of steps on this problem. Make sure you show that if a and b are in G that $a * b \in G$, i.e. that it does **not** equal -1.)

(2) (**generalized socks and shoes**)

Let $g_1, \ldots, g_n$ be elements of a group. Show that

$$(g_1 \cdots g_n)^{-1} = {g_n}^{-1} \cdots {g_1}^{-1}.$$

(3) Show that the set of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

(with $a, b, c \in R$) forms a group under matrix multiplication. Make a table of this group when $R = \mathbb{Z}_2$. Does this group look familiar?

(4) In this problem we deal with quaternions.
  (a) Show for quaternions q_1 and q_2 that $\overline{q_1 q_2} = \overline{q}_2 \, \overline{q}_1$ (notice the reverse! One could prove this relation by "brute force" but there are slicker ways).
  (b) Recall from homework 1 problem 3 that a quaternion is in $\mathbb{S}^3$ if and only if $\overline{q} = q^{-1}$. Use this fact, socks and shoes, and the first part of this problem to show that $\mathbb{S}^3$ is closed under multiplication.
  (c) Show that $\mathbb{S}^3$ is a group. (the only thing that you need to do at this point is show that the inverse of an element in $\mathbb{S}^3$ is also in $\mathbb{S}^3$).

(5) Show that the following are equivalent for elements g and h in a group (NONE ARE TRUE BY THEMSELVES!! PROVE THIS BY FIRST ASSUMING A PART IS TRUE AND USE THAT TO SHOW THAT ANOTHER PART IS TRUE!)
  (a) $(g * h)^{-1} = g^{-1} * h^{-1}$
  (b) $g^2 * h^2 = (g * h)^2$
  (c) $g * h * g^{-1} * h^{-1} = e$
  (d) g and h commute.

(6) (a) Prove by hand that for matrices $A \in M_2(R)$ and $B \in M_2(R)$ (i.e. A and B are 2 by 2 matrices with coefficients in R) that

$$\det(AB) = \det(A)\det(B).$$

  (b) Use the first part of this problem to show that

$$\{A \in M_2(R) : \det(A) = 1\}$$

forms a group under matrix multiplication. It is called the 2 by 2 *special linear group over* R and is denoted $\mathrm{SL}(2, R)$.

(c) Show that $\mathrm{SL}(2, R)$ is finite if and only if R is finite.

(d) Make a multiplication table of $\mathrm{SL}(2, \mathbb{Z}_2)$. Does table look like a table we have already seen?

(e) **Research**

- Now for matrices $A \in M_n(R)$ and $B \in M_n(R)$ it turns out that one still has that $\det(A)\det(B)$. Cite a reference which proves this fact.
- Use the previous fact to show that $\mathrm{SL}(n, R)$ (the set of n by n matrices with coefficients in R and determinant 1) is closed under matrix multiplication.
- Look up the formula for the *adjoint* of a matrix. Show that the adjoint of a matrix with coefficients in R still has coefficients in R (that is $M_n(R)$ is closed under taking adjoints).
- Reference a source which proves that for a matrix A that
$$A^*A = AA^* = \det(A)\, I_n.$$
- Use these facts to show that the inverse of an element in $\mathrm{SL}(n, R)$ is still in $\mathrm{SL}(n, R)$ and conclude that this more general set is also a group.

(7) Let $(G, *)$ and $(H, \bullet)$ be groups.

(a) Prove that $G \times H$ with the binary operation
$$(g, h) \cdot (g', h') := (g * g', h \bullet h')$$
forms a group.

(b) Show that with the operation defined above, $G \times H$ is abelian if and only if G and H are abelian.

(c) Show that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has the same multiplication table as the set of symmetries of a rectangle which is *not* a square.