HOMEWORK 2

(1) Let $(G, *)$ and $(H, \bullet)$ be groups. Prove that $G \times H$ with the binary operation $(g, h) \cdot (g', h') := (g * g', h \bullet h')$ forms a group.

(2) Show that the following are equivalent for elements g and h in a group.
   (a) $(g * h)^{-1} = g^{-1} * h^{-1}$
   (b) $g^2 * h^2 = (g * h)^2$
   (c) $g * h * g^{-1} * h^{-1} = e$
   (d) g and h commute.

(3) Let $f : X \longrightarrow Y$ be a function. Show that $\{g \in S_X : f \circ g = g\}$ forms a group under composition (recall that S_X is the set of invertible function of X).

(4) Let A be a subset of a set X. Show that $\{g \in S_X : g(A) = A\}$ forms a group under composition.

(5) For any two sets A and B define $A \Delta B$ by $(A \cup B) - (A \cap B)$. Let X be any set and consider the power set $\mathcal{P}(X) := \{A : A \subseteq X\}$. Show that $(\mathcal{P}(X), \Delta)$ forms a group. Is this group abelian? Write down the multiplication table in the special case that $X = \{a, b\}$.

(6) Show that the set of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

(with $a, b, c \in \mathbb{Z}$) forms a group under matrix multiplication called the (or *a* since one needn't take a, b, c to live in the integers to get a group, one could take $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ or any other so called *ring* as well) Heisenberg group. Is this group abelian?

(7) Let g be an element of a group. Show that $(g^{-1})^{-1} = g$ (**hint** use the uniqueness of inverses).

(8) Let G be a group of even order. Show that G contains a non-identity element whose square is the identity.

(9) Show that set of n by n matrices of determinant 1 with integer entries (again entries could lie in any ring) forms a group called the *special linear group* and denoted $\mathrm{SL}(n, \mathbb{R})$ (hint, you may wish to use the fact that $\det(AB) = \det(A)\det(B)$).