

### HOMEWORK 3

- (1) The following exercise deals with the so called normalizer of a subgroup which is defined as  $N_G(H) := \{g \in G \mid ghg^{-1} \in H \text{ for all } h \in H\}$  for a subgroup  $H$  of a group  $G$ .
- (a) Show  $N_G(H)$  is also a subgroup of  $G$  (bonus, find an example where this does not hold if  $G$  is infinite).
  - (b) Show that  $H$  is normal in  $N_G(H)$ .
  - (c) Show that  $N_G(H)$  is the largest subgroup of  $G$  with  $H$  normal in  $N_G(H)$  in the sense that if  $K$  is a subgroup of  $G$  and  $H$  is normal in  $K$ , then  $K \subseteq N_G(H)$ .
  - (d) Let  $G = S_3$  and  $H$  be the subgroup generated by the bijection which exchanges 1 and 2. Find the normalizer of  $H$  in  $S_3$ .
- (2) In the following problem let  $H$  and  $N$  be subgroups of a group  $G$ .
- (a) Show that the intersection of  $N$  and  $H$  is a subgroup of  $G$ . Also show that it is the largest subgroup of  $G$  which is contained in both  $H$  and  $N$ .
  - (b) Suppose in addition that  $N$  is normal in  $G$ . Then show

$$HN = \{hn \mid h \in H, n \in N\}$$

is also a subgroup of  $G$ . Also show that it is the smallest subgroup of  $G$  which contains both  $H$  and  $N$ .

- (3) This problem deals with the subset

$$SO(2, \mathbb{R}) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

of  $M(2, \mathbb{R})$ .

- (a) Show that  $SO(2, \mathbb{R})$  is an abelian subgroup of  $SL(2, \mathbb{R})$  where the operation is matrix multiplication (you might need to look up some trig identities).
  - (b) Is this subgroup isomorphic to any of the groups  $\mathbb{Q}^*$ ,  $\mathbb{R}^*$ , or  $\mathbb{C}^*$  under multiplication? Explain why or why not.
  - (c) Give a geometric interpretation of this group.
  - (d) Show that it is not a normal subgroup of  $SL(2, \mathbb{R})$ .
- (4) Suppose that  $g, h \in G$  are two commuting elements which generate subgroups of  $G$  which meet only at the identity.
- (a) Show that the order of  $gh$  is the least common multiple of the orders of  $g$  and  $h$ .
  - (b) Show that both hypothesis on  $g$  and  $h$  are necessary (part of your grade in this problem includes figuring out the logic of exactly what this means and how to prove without doing either too much work or too little work).