HOMEWORK 2

(1) Let $A = \{1,2,3,4\}$, $B = \{a,b,c\}$, $C = \{x,y,z,w\}$, and $D = \{u,v,t\}$. Consider the relations $R : A \longrightarrow B$ given by $\{(1,a),(1,b),(2,a),(3,c)\}$, $S : B \longrightarrow C$ given by $\{(a,y),(b,y),(c,w)\}$, and $T : C \longrightarrow D$ given by $\{(x,u),(x,t),(w,v)\}$.
   (a) Which of R, S and T are injective, surjective?
   (b) Which are single valued?
   (c) Which are functions?
   (d) Compute $S \circ R$.
   (e) Compute $R^{-1} \circ S^{-1}$.
   (f) Compute the image and domain of T.
   (g) Verify that $(T \circ S) \circ R = T \circ (S \circ R)$.

(2) If $R : A \longrightarrow B$ and $R^{-1} : B \longrightarrow A$ are functions, show that $R \circ R^{-1} = Id_B$ and $R^{-1} \circ R = Id_A$.

(3) Define a relation $R : \{1,2,3\} \longrightarrow \{a,b,c\}$ so that $R^{-1} \circ R \neq Id_A$.

(4) How many relations are there from $\{a,b,c\}$ to itself? How many functions? How many bijections?

   Sol'n

   Relations from A to B are just subsets of $A \times B$. $\{a,b,c\} \times \{a,b,c\}$ has $3 \cdot 3 = 9$ elements. Thus there are $2^9 = 512$ relations from A to B. For any function there are 3 choices were to send the three elements of the first set. Thus there are $3 \cdot 3 = 9$ functions. For a bijection there are three choices where to send a then there are two choices left where to send b and then only one choice left as to where to send c. Thus there are a total of 6 bijections.

(5) Show that the composite of injective (surjective) relations is injective (surjective).

   Sol'n

   Let $R : A \longrightarrow B$ and $S : B \longrightarrow C$ be two injective relations and assume that (a,c) and (a',c) are in $S \circ R$. We need to show that $a = a'$. Then there exists a b and a b' in B with

$$(a,b) \in R \text{ and } (b,c) \in S$$

   while

$$(a',b') \in R \text{ and } (b',c) \in S.$$

   Since S is one-to-one, this implies that $b = b'$. Hence

$$(a,b) \text{ and } (a',b) \in S.$$

   Since S is also injective, this guarantees us that $a = a'$, as desired.

   Now assume $R : A \longrightarrow B$ and $S : B \longrightarrow C$ are two surjective relations and let $c \in C$. We need to find an $a \in A$ with $(a,c) \in S \circ R$. But since S is surjective there is a $b \in B$ with $(b,c) \in S$. Moreover since R is surjective there is an $a \in A$ with $(a,b) \in R$. Glueing these two facts together gives us that $(a,c) \in S \circ R$ as desired.

(6) Let $A := \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define a relation on A via $(a,b)R(c,d)$ if and only if $ad - bc = 0$.
  (a) With out making reference to rational numbers that may not be integers, prove that R is an equivalence relation (i.e. do the proof without using any division).
  (b) Show that the function

$$A \longrightarrow \mathbb{Q}$$
$$(a,b) \mapsto a/b$$

  induces a bijection from A/R to $\mathbb{Q}$.

(7) Let R be the relation on $A := \{1,2,3,4,5,6\}$ given by

$$\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,4),(4,1),(1,3),(3,1),(3,4),(4,3),(2,5),(5,2)\}$$

  (a) Draw a digraph which verifies that R is an equivalence relation.
  (b) Calculate $1/R$, $2/R$, $3/R$, $4/R$, $5/R$, and $6/R$.
  (c) How many elements are there in A/R?

(8) (Experiment!)
  (a) Play with some simple functions from $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ (such as $(s,t) \mapsto (s^3+t^2, st^2, s^3+t)$). Does it seem easy in general to figure out whether or not such a function is injective?
  (b) Show that the function $(s,t) \mapsto (2s+t, s+2t, s+t)$ is injective. What makes this easier?

  Sol'n
  Let us call the function in question F. Suppose that $F(s,t) = F(s',t')$. Then this means that $2s+t = 2s'+t'$, $s+2t = s'+2t'$, and $s+t = s'+t'$. Subtracting the second equation from the third gives us that $t = t'$. Now going back to the third equation $s+t = s'+t' = s'+t$ implies that $s = s'$. Hence $(s,t) = (s',t')$ and F is one-to-one.

(9) Again let $R : A \longrightarrow B$ and $S : B \longrightarrow C$ be relations.
  (a) Show that

$$\operatorname{dom}(S \circ R) = \operatorname{dom}(R) \cap R^{-1}(\operatorname{dom}(S)).$$

  Sol'n
  $\subset$ Let $a \in \operatorname{dom}(S \circ R)$. Then there exists a $c \in C$ so that $(a,c) \in S \circ R$. This means there is a $b \in B$ so that $(a,b) \in R$ and $(b,c) \in S$. Hence $a \in \operatorname{dom}(R)$ and $b \in \operatorname{dom}(S)$. It remains to show that a is in the inverse image of the domain of S. But since $(a,b) \in R$ we have that a is related to some element in the domain of S. That is

$$a \in R^{-1}\operatorname{dom}(S)\,[1]$$

  as well.

[1] Recall that $S^{-1}\left(\overline{B}\right) := \{a : (a,b) \in R \text{ for some } b \in \overline{B}\}$

$\supset$ Let
$$a \in \text{dom}(R) \cap R^{-1}(\text{dom}(S)).$$
Then in particular $a \in R^{-1}(\text{dom}(S))$. That is there exists a $b \in \text{dom}(S)$ so that $(a,b) \in R$. Since $b \in \text{dom}(S)$ there exists a $c \in C$ so that $(b,c) \in S$. But this guarantees us that $(a,c) \in S \circ R$. That is $a \in \text{dom}(S)$ as desired.

(b) Now assume that the set A equals the set C and that $R \circ S = Id_B$ and $S \circ R = Id_A$. Use the previous part to show that $\text{dom}(R) = A$ and $\text{dom}(S) = B$.

Sol'n
Since $\text{dom}(Id_A) = A$ and $S \circ R = Id_A$ by assumption, we know from the previous part that
$$A = \text{dom}(Id_A) = \text{dom}(S \circ R) = \text{dom}(R) \cap R^{-1}(\text{dom}(S)) \subset \text{dom}(R) \subset A.$$
(the last inclusion is because the domain of any relation from A to B is contained in A by definition) we know that all inclusions are really equalities. In particular $\text{dom}(R) = A$. The proof that $\text{dom}(S) = B$ is completely symmetric.

(c) Under the assumptions from part (9b) show that both R and S are single valued. (This part takes a lot of focus and understanding of the definitions!)

Sol'n
Suppose that (a,b) and (a,b') are in R. We need to show that $b = b'$. By the previous part we know that b and b' are in the domain of S. Hence there is a $\overline{a}, \overline{a}' \in A$ so that
$$(b,\overline{a}), (b',\overline{a}') \in S. \tag{1}$$
But since
$$(a,b) \in R \text{ and } (b,\overline{a}) \in S \Rightarrow (a,\overline{a}) \in S \circ R \ (= Id_A)$$
and
$$(a,b) \in R \text{ and } (b,\overline{a}') \in S \Rightarrow (a,\overline{a}') \in S \circ R \ (= Id_A)$$
But by assumption $S \circ R = Id_A$. This means that both $(a,\overline{a})$ and $(a,\overline{a}')$ are equal (a,a), i.e. $a = \overline{a} = \overline{a}'$. Plugging this into equation (1) yields
$$(b,a), (b',a) \in S.$$
Since $(b',a) \in S$ and $(a,b) \in R$ we have that $(b,b') \in R \circ S$. But again $R \circ S = Id_B$ by assumption. Hence $b = b'$. Therefore R is single valued. Again, the problem is completely symmetric (i.e. if one replaces A by B and R by S we have exactly the same statement) we also see that S is single valued.

(d) Under the assumptions from part (9b) show that both S and R are functions (you have already shown this in the other parts of the problem) and $S = R^{-1}$.

Sol'n

In part (9b) we showed that R and S domains are equal to A and B respectively. Moreover, part (9c) proved that both relations in questions were single valued. By definition this means that both are functions. For the second claim we need to show that $R^{-1} = S$.

- $\subset$ Let $(b, a) \in S$. Since the domain of R is A we know that (a, b') is in R for some $b' \in B$. But this means that $(b, b') \in R \circ S$. But since this latter set is Id_B we know that $b = b'$. Hence $(b, a) = (b', a) \in R^{-1}$.
- $\supset$ Conversely suppose that $(b, a) \in R^{-1}$. Then $(a, b) \in R$. Since the domain of S is B we know that $(b, a') \in S$ for some $a' \in A$. Hence $(a, a') \in S \circ R = Id_A$ which imples that $a = a'$. Thus $(b, a') = (b, a)$ and therefore $(b, a) \in S$ as desired.