HOMEWORK 2

(1) Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{x, y, z, w\}$, and $D = \{u, v, t\}$. Consider the relations $R : A \longrightarrow B$ given by $\{(1, a), (1, b), (2, a), (3, c)\}$, $S : B \longrightarrow C$ given by $\{(a, y), (b, y), (c, w)\}$, and $T : C \longrightarrow D$ given by $\{(x, u), (x, t), (w, v)\}$.
   (a) Which of R, S and T are injective, surjective?
   (b) Which are single valued?
   (c) Which are functions?
   (d) Compute $S \circ R$.
   (e) Compute $R^{-1} \circ S^{-1}$.
   (f) Compute the image and domain of T.
   (g) Verify that $(T \circ S) \circ R = T \circ (S \circ R)$.

(2) If $R : A \longrightarrow B$ and $R^{-1} : B \longrightarrow A$ are functions, show that $R \circ R^{-1} = Id_B$ and $R^{-1} \circ R = Id_A$.

(3) Define a relation $R : \{1, 2, 3\} \longrightarrow \{a, b, c\}$ so that $R^{-1} \circ R \neq Id_A$.

(4) How many relations are there from $\{a, b, c\}$ to itself? How many functions? How many bijections?

(5) Show that the composite of injective (surjective) relations is injective (surjective).

(6) Let $A := \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define a relation on A via $(a, b)R(c, d)$ if and only if $ad - bc = 0$.
   (a) With out making reference to rational numbers that may not be integers, prove that R is an equivalence relation (i.e. do the proof without using any division).
   (b) Show that the function

$$A \longrightarrow \mathbb{Q}$$
$$(a, b) \mapsto a/b$$

   induces a bijection from A/R to $\mathbb{Q}$.

(7) Let R be the relation on $A := \{1, 2, 3, 4, 5, 6\}$ given by

$$\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (1, 3), (3, 1), (3, 4), (4, 3), (2, 5), (5, 2)\}$$

   (a) Draw a digraph which verifies that R is an equivalence relation.
   (b) Calculate $1/R$, $2/R$, $3/R$, $4/R$, $5/R$, and $6/R$.
   (c) How many elements are there in A/R?

(8) (Experiment!)
   (a) Play with some simple functions from $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ (such as $(s, t) \mapsto (s^3 + t^2, st^2, s^3 + t))$. Does it seem easy in general to figure out whether or not such a function is injective?
   (b) Show that the function $(s, t) \mapsto (2s + t, s + 2t, s + t)$ is injective. What makes this easier?

(9) Again let $R : A \longrightarrow B$ and $S : B \longrightarrow C$ be relations.

(a) Show that

$$\operatorname{dom}(S \circ R) = \operatorname{dom}(R) \cap R^{-1}(\operatorname{dom}(S)).$$

(b) Now assume that the set A equals the set C and that $R \circ S = Id_B$ and $S \circ R = Id_A$. Use the previous part to show that $\operatorname{dom}(R) = A$ and $\operatorname{dom}(S) = B$.

(c) Under the assumptions from part (9b) show that both R and S are single valued. (This part takes a lot of focus and understanding of the definitions!)

(d) Under the assumptions from part (9b) show that both S and R are functions (you have already shown this in the other parts of the problem) and $S = R^{-1}$.