

HOMEWORK 2

- (1) Write down an associative and a non-associative multiplication table with entries in $\{a, b, c\}$ (prove they are associative or not, but do you really want to do 27 calculations?).

Proof. Nobody seemed to have any problems with this problem. □

- (2) Let $G := \mathbb{R} - \{-1\}$ and define $a * b := a + b + a \cdot b$ (here $+$ and \cdot have their usual meanings). Show that $(G, *)$ is a group.

Proof. To show something is a group we simply need to verify the 3 properties as well as show that $*$ is a binary operation.

- 0) *** is a binary operation** We must show that $*$ is a function from $G \times G \rightarrow G$. The important part is that we have to show that if $a \in G$ and $b \in G$ that $a * b \in G$. Thus we suppose that $a * b = -1$ (and arguing by contraposition) show that either $a = -1$ or $b = -1$). Then

$$\begin{aligned} a * b &= ab + a + b = -1 \\ &\Rightarrow ab + a + b + 1 = 0 \\ &\Rightarrow a(b + 1) + b + 1 = 0 \\ &\Rightarrow (a + 1)(b + 1) = 0 \\ &\Rightarrow a = -1 \text{ or } b = -1. \end{aligned}$$

- 1) **Associative** We derive formulas for both $(a * b) * c$ and $a * (b * c)$ and inspection of both will show that both sides are equal.

$$\begin{aligned} (a * b) * c &= (ab + a + b) * c \\ &= (ab + a + b)c + ab + a + b + c \\ &= abc + ac + bc + ab + a + b + c. \end{aligned}$$

On the other hand,

$$\begin{aligned} a * (b * c) &= a * (bc + b + c) \\ &= a(bc + b + c) + a + (bc + b + c) \\ &= abc + ab + ac + a + bc + b + c \\ &= abc + ac + bc + ab + a + b + c. \end{aligned}$$

- 2) **Identity**¹ I claim that 0 is the identity of G . First note that $0 \neq -1$ so that $0 \in G$. Now

$$\begin{aligned} 0 * a &= 0a + 0 + a \\ &= a. \end{aligned}$$

(We do not need to check the other direction, why?)
Thus $(G, *)$ has an identity.

¹Many students solved for the identity, I could care less if a little green man told you what the identity is, all I care about is that some element of G satisfies the properties of being an identity

- 3) **Inverse** Let $a \in G$. I claim that $\frac{-a}{1+a} \in G$ (this fraction makes sense since $a \in G$, i.e. $a \neq -1$) and

$$a * \left(\frac{-a}{1+a} \right) = \left(\frac{-a}{1+a} \right) * a = 0.$$

And since we showed in the last step that 0 is the identity of G , this will prove that $a^{-1} = \left(\frac{-a}{1+a} \right)$ (in particular, a^{-1} exists).

- Suppose that $\frac{-a}{1+a} = -1$. Then $-a = -a - 1 \Rightarrow 0 = -1$, a contradiction.
- This is just another calculation ²

$$\begin{aligned} a * \left(\frac{-a}{1+a} \right) &= a \left(\frac{-a}{1+a} \right) + a + \left(\frac{-a}{1+a} \right) \\ &= a \left(\frac{-a}{1+a} \right) + a \left(\frac{1+a}{1+a} \right) + \left(\frac{-a}{1+a} \right) \\ &= \frac{-a^2 + a + a^2 - a}{1+a} \\ &= 0. \end{aligned}$$

(Again, we don't need to check the other direction, why?) □

- (3) Let $\mathbb{H} := \mathbb{R}^4$ (which we think of as $\mathbb{R} \times \mathbb{R}^3$). Usual facts about dot and cross products from calc III to show that \mathbb{H} equipped with the binary operation

$$(a, \vec{v}) * (b, \vec{w}) := (ab - \vec{v} \cdot \vec{w}, ab\vec{v} + a\vec{w} + \vec{v} \times \vec{w})$$

is associative and has an identity (here \cdot and \times denote the usual dot product and cross products in \mathbb{R}^3).

Proof. This is a long uninspiring calculation. The most straightforward way is by using components. If you think I'm going to write this all out in Latex you guys are all crazy! :). □

- (4) Let g be an element of a group. Show that $(g^{-1})^{-1} = g$.

Proof. By uniqueness of inverses, we only need to show that g behaves like the inverse of g^{-1} . That is we need to show that $g * g^{-1} = g^{-1} * g = e$, but this is what is required for g^{-1} to be the inverse of g ! □

²Again, I don't care where you get this from, I only care that it works!