HOMEWORK 12

In this homework set R will denote an arbitrary ring.

(1) Consider the set $C(\mathbb{R})$ of continuous functions from the real numbers to itself.

(a) Use the basic facts about continuous functions to show that $C(\mathbb{R})$ is a commutative ring with identity when the addition is taken as $(f+g)(x) := f(x)+g(x)$ and the multiplication is defined as $(f\cdot g)(x) := f(x)g(x)$. Show that it is however not an integral domain (just draw graphs).[1]

(b) Use the basic facts about continuous functions to show that $C(\mathbb{R})$ is a non-commutative ring with identity when the addition is taken as $(f+g)(x) := f(x)+g(x)$ and the multiplication is defined as $(f\cdot g)(x) := f(g(x))$ (i.e. composition of functions).

(c) Let $T \subset \mathbb{R}$. Show directly that the set

$$I_T := \{f \in C(\mathbb{R}) : f(t) = 0 \text{ for all } t \in T\}$$

is an ideal of $C(\mathbb{R})$ when using the multiplication from 1a but it is not an ideal in general when we use the multiplication from 1b.

(d) For an $a \in \mathbb{R}$ define a function

$$\begin{aligned} ev_a : C(\mathbb{R}) &\longrightarrow \mathbb{R} \\ f &\mapsto f(a). \end{aligned}$$

Show that the function ev_a is a surjective homomorphism of rings when $C(\mathbb{R})$ is given the multiplication from 1a but that it is not a homomorphism when given the multiplication from 1b. Express the kernel of this function in terms of the notation from 1c.

(2) Suppose that R has an identity, 1_R. Show that $1_R = 0_R$ if and only if R contains only one element.

(3) Let R be any ring. Let $M_2(R)$ be the set of matrices of the form

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

with a, b, c and d in R and equipped with matrix addition and multiplication, the latter being defined as

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ d' & e' \end{pmatrix} := \begin{pmatrix} aa' + bd' & ab' + be' \\ da' + ed' & db' + ee' \end{pmatrix}.$$

(a) Show that $M_2(R)$ is a ring (yes, you need to verify that matrix multiplication is associative).

(b) Show that if R has an identity, then so does $M_2(R)$ and if $1_R \neq 0_R$, then $M_2(R)$ is not commutative. [2]

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[1] if we instead looked at analytic functions from $\mathbb{R}$ to itself (i.e. functions that can locally be expressed as a power series), this set **would** form an integral domain

[2] take a little care here since statements like $1_R + 1_R = 2$ are nonsense in an abstract ring

(c) Suppose furthermore that R is commutative. Show that a matrix is invertible if and only if its determinant (defined by the same formula as usual) is invertible in R.

(d) For concreteness fix $R = \mathbb{Z}$. Show that the set of upper triangular matrices, i.e. matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

forms a sub-ring, $UT = UT_2(\mathbb{Z})$ of $M_2(\mathbb{Z})$ which is not an ideal. Show that the set of matrices of the form

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

is an ideal of UT.

(4) Let R be a ring with identity, 1_R. Show that the set

$$U(R) := \{r \in R : \text{there exists } s \in R \text{ so that } rs = 1_R = sr\}$$

equipped with the multiplication coming from the ring's multiplication forms a group. Name two special cases of this construction that we have encountered already (state the homework number and problem number).

(5) Let d be a real number which is square-free (i.e., no integer, which is a different integer squared, divides d). And let R denote either $\mathbb{Z}$ or $\mathbb{Q}$.

(a) Show that in either case the set

$$R[\sqrt{d}] := \{a + b\sqrt{d} : a, b \in R\}$$

forms an integral domain.

(b) Show that it is a field if R is taken to be $\mathbb{Q}$ (and not $\mathbb{Z}$).

(c) Calculate $U(\mathbb{Z}[i])$ (think carefully what it means for a $z \in \mathbb{Z}[i]$ to have $1/z \in \mathbb{Z}[i]$ as well).