# HOMEWORK 11

As usual $G$ and $H$ are assumed to be groups.

(1) We explore a particular action.

(a) Show that $G$ acts on its set of subgroups

$$g \bullet H := gHg^{-1}.$$

for all subgroups $H < G$.

(b) Show that the stabilizer of a subgroup $H$ is its normalizer:

$$\{g \in G : gHg^{-1} = H\}$$

(this is just about by definition).

(c) Verify the orbit stabilizer theorem for this action in the special case that $G = S_4$ and $H = \langle (1,2,3,4), (1,3) \rangle$ (do this by showing that the normalizer of this subgroup has order 8 and the orbit of this subgroup has order 3).

(d) Explain how the 3 conjugate subgroups of $H$ that you found in the previous part can be thought of as labelings of the vertices of a square. (yes, this question is vague, but please try to write down something).

(e) Let $H$ be as above, by labeling the conjugate subgroups of $H$ as $x, y$ and $z$ write down the explicit homomorphism

$$S_4 \longrightarrow \mathrm{Bij}(\{x, y, z\}) \cong S_3$$

(you can express this homomorphism by simply telling me where the generators $(1,2), (2,3)$ and $(3,4)$ go).

(2) Let $H$ and $K$ be subgroups of a group $G$ and consider the set

$$HK := \{hk : h \in H and k \in K\}$$

(recall that in class we showed this was also a subgroup of $G$ in the special case that either $H$ or $K$ is normal in $G$.)

(a) Let $G := S_3$ and $H = \langle (1,2) \rangle$ and $K = \langle (2,3) \rangle$. Show that $HK$ consists of 4 elements and is not a subgroup of $S_3$.

(b) Show that $H \times K$ acts on $HK$ via $(h,k) \bullet x := hxk^{-1}$.

(c) Show that this action is transitive (i.e. that the action has only one orbit).

(d) Show that the stabilizer of $e_G = e_H e_K \in HK$ is in bijection with $H \cap K$.

(e) Use the orbit stabilizer theorem to conclude that

$$|HK| = \frac{|H|\,|K|}{|H \cap K|}.$$

(3) In this problem we work out some conjugacy classes.

(a) Write down a representitive of the 11 conjugacy classes of $S_6$ along with the number of elements in each of these conjugacy classes. Verify that the sum of these numbers equals $6! = 720$.

(b) Do the same for the group $\langle e^{\frac{\pi i}{4}}, j \rangle$ considered in homework 9.

   (c) Do the same for $S_5$
(4) Let $G$ act on a set $X$.
   (a) Show that
$$\text{Stab}(g \bullet x) = g\text{Stab}(x)g^{-1}.$$
   (this formula says, among other things, that stabilizers of different points in the same orbit are isomorphic).
   (b) Consider the action of $S_4$ on three dimensional cube discussed in class (a fancy way of thinking of this action is to label the 8 three cycles in $S_4$ by the numbers 1 through 8. Then have $S_4$ act on this set of 3 cycles via conjugation. This action is the same as the action of $S_4$ on the set of 8 vertices of the cube) Verify the formula just derived for this action by comparing the stabilizer of the 6 different faces of the cube.
(5) (**BONUS!**) In this problem we show that $A_5$ is simple. For an element $g$ of a group $G$ define $cc_G(g)$ to equal the number of elements in $g's$ conjugacy class.
   (a) (**tricky!**) For an element $\sigma \in A_n$ show that $cc_{A_n}(\sigma) = cc_{S_n}(\sigma)$ if and only if there is an odd permutation which centralizes $\sigma$. Show if the two numbers are not the same, then the second number is twice the first number.
   (b) Write down representatives of the 5 conjugacy classes of $A_5$ (be careful! even though $(1, 2, 3, 4, 5)$ and $(1, 2, 3, 5, 4)$ are conjugate in $S_5$ they are not conjugate in $A_5$.) Compare these numbers to the number of faces, edges, and vertices of a dodechahedron.
   (c) Show that a proper subgroup of $A_5$ cannot be the union of conjugacy classes (add up the numbers from the previous part and show they can not possibly add up to a divisor of 60).
   (d) Conclude that $A_5$ is simple (normal subgroups are unions of conjugacy classes!).