

HOMEWORK 11

As usual G and H are assumed to be groups.

- (1) Show explicitly that any two Sylow p -groups of a group are isomorphic.
- (2) In this problem we deal with the explicit example of S_4 .
 - (a) Find all Sylow 2-groups of S_4 and show explicitly that they are conjugate. Which Sylow's theorem does this verify? (This is a boring, straight forward calculation, one of the few that I've asked you to do for homework!)
 - (b) How many Sylow 2-groups are there in S_4 ? Explain in detail why this verifies Sylow's third theorem.
 - (c) What group is a Sylow 2-group of S_4 isomorphic to?
 - (d) Without doing any more calculations, show that quaternion group of order 8 is not a subgroup of S_4 .
 - (e) Show that quaternion group of order 8 is not a subgroup of S_7 by first showing that a Sylow 2-group of S_7 is isomorphic to $D_4 \times \mathbb{Z}_2$ and then counting the number of elements of order 4.
- (3) Show that there are non-trivial normal subgroups of any group of the following orders:
 - (a) 42
 - (b) 200
 - (c) 255

- (4) Suppose that H and K subgroups of a finite group G and consider the set

$$HK := \{hk : h \in H \text{ and } k \in K\}$$

(recall that this set is a group whenever H is normal in G but not necessarily otherwise). Show that $(h, k) \bullet x := h x k^{-1}$ gives us a transitive action of $H \times K$ on HK with stabilizer $H \cap K$. Conclude from the orbit-stabilizer theorem that

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

(the fact that this is true whenever H is normal has been previously established in homework 9 problem 11).

- (5)
 - (a) Suppose that a group G has a normal p -group P and let Q be any q -group ($p \neq q$) of the same group G . Show that G contains a subgroup of order pq .
 - (b) Use this fact and the fact that A_4 contains no subgroup of order 6 to show that A_4 contains no normal subgroups of orders 2 or 3.
- (6) Let G be a group of order $30 = 5 \cdot 3 \cdot 2$ with P_5 , P_3 and P_2 Sylow 5, 3 and 2 groups resp.
 - (a) Show that if P_5 is not normal that there are exactly 6 Sylow 5-groups and 24 elements of order 5 (4 from each Sylow 5-group).
 - (b) Show that if P_3 is not normal that there are exactly 10 Sylow 3-groups and 20 elements of order 3 (2 from each Sylow 3-group).
 - (c) Conclude that either P_3 or P_5 is normal.

- (d) Finally conclude that G contains a subgroup of order 15.
- (7) Use a similar logic as what was used in the previous problem to show that any group of order 105 contains a subgroup of order 35.
- (8) In this set of problems we explore groups of order 375. As such let G be an arbitrary group of order 375 and let P_5 and P_3 be the obvious Sylow p -groups of G .
 - (a) Show that P_5 is normal in G .
 - (b) Show that $g \bullet h := ghg^{-1}$ ($g \in P_3$ and $h \in P_5$) gives us a well defined action of P_3 on P_5 .
 - (c) Use the fact that the order of P_5 is 125 (which is equal to 2 mod 3) to conclude that there exists a global fixed point $z \in P_5$ with $z \neq 1_G$.
 - (d) Show that G contains a subgroup of order 15, namely the group generated by z and P_3 .
 - (e) Conclude that G contains an element of order 15 (**hint** what does every group of order 15 look like?)
 - (f) (**Challenge!**) Show that we can assume that z lives in the center of G (**hint** break this into two cases, one where P_5 not abelian and the case where P_5 is abelian and then use the fact that the center of P_5 has order 5 in the first case and order 125 in the second).