HOMEWORK 10

As usual G and H are assumed to be groups.

(1) Let G_1, G_2 be groups.
   (a) Show that the map
$$\pi_1 : G_1 \times G_2 \to G_1$$
$$(g_1, g_2) \mapsto g_1$$
   and
$$\pi_2 : G_1 \times G_2 \to G_2$$
$$(g_1, g_2) \mapsto g_2$$
   are surjective homomorphisms.
   (b) Show that $\ker(\pi_1) \cong G_2$ and $\ker(\pi_2) \cong G_1$.
   (c) Conclude that $(G_1 \times G_2)/G_2 \cong G_1$ (really we are not modding out by G_2 here, but only a subgroup of the direct product isomorphic to G_2).

(2) In this problem we cover a primitive way of showing that certain groups are *idecomposable*, i.e. not (isomorphic to) the direct product of two non-trivial groups.
   (a) Show that the direct product of two abelian groups is abelian.
   (b) Use the first part of this problem and the classification of groups of small order to show that the groups Q_8, D_4, D_5, A_4 and S_3 are indecomposable.

(3) In this problem, we show that $D_6 \cong D_3 \times \mathbb{Z}_2$.
   (a) Show that $H := \langle a^2, b\rangle$ is a normal subgroup of D_6 which is isomorphic to D_3 (or S_3).
   (b) Show that $K := \langle a^3\rangle$ is a normal subgroup of D_6 which is isomorphic to $\mathbb{Z}_2$.
   (c) Show that $H \cap K = \{e\}$.
   (d) Show that $G = \langle H \cup K\rangle$ (**hint** what is the only divisor of 12 larger than 6?).
   (e) Conclude from the direct product recognition theorem that $D_6 \cong D_3 \times \mathbb{Z}_2$.

(4) In this problem we assume that G is a finite group whose every non-identity element has order 2.
   (a) Show that G is abelian. (**hint** For any two elements, $a, b \in G$ we have that $e = (ab)^2$).
   (b) If $X := \{x_1, \ldots x_n\}$ is a minimal generating set for G (i.e., no proper subset of X generates G) show that
$$(\epsilon_1, \ldots, \epsilon_n) \mapsto {x_1}^{\epsilon_1}, \ldots {x_n}^{\epsilon_n}$$
   defines an isomorphism from
$${\mathbb{Z}_2}^n \longrightarrow G.$$
   (c) Conclude that any group whose order is not a power of 2 contains an element whose order is not 2.

(5) Let T be a subset of $\{1, \ldots, n\}$.
  (a) Show that the set of permutations $\sigma \in S_n$ which satisfy $\sigma(T) = T$ is a subgroup of S_n which we denote by S_T.[1]
  (b) Let $T^c := \{1, \ldots, n\} \setminus T$ (the complelement of T). Show that $S_{T^c} = S_T$ (i.e. the two groups are equal, not just isomorphic). (**hint** I am just asking you to show that a bijection fixes a set if and only if it fixes its compliment).
  (c) Show that $S_T \cong S_m \times S_{n-m}$ where m is the number of elements in T. (**hint**, consider the subgroup of S_T defined as the set of permutations, σ, which satisfy $\sigma(i) = i$ for every $i \in T^c$.)

(6) Suppose that H_1 is a normal subgroup of G_1 and H_2 is a normal subgroup of G_2.
  (a) Show that $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$.
  (b) By using the first homomorphism theorem, show that

$$(G_1 \times G_2) \,/\, (H_1 \times H_2) \cong G_1/H_1 \times G_2/H_2$$

(7) In this problem, we explore yet another way of building new groups from old called the fiber product. The relevant ingredients are three groups G_1, G_2 and H and two homomorphisms $\varphi_i : G_i \to H$ $(i = 1, 2)$. We denote the pair (φ_1, φ_2) by φ. Define

$$G_1 \times_\varphi G_2 := \{(g_1, g_2) \in G_1 \times G_2 : \varphi_1(g_1) = \varphi_2(g_2)\}\,.$$

$G_1 \times_\varphi G_2$ is called the *fiber product* of G_1 and G_2 over φ. (**In parts (c)-(e) we assume that φ_1 is surjective**)
  (a) Show that $G_1 \times_\varphi G_2$ is a subgroup of $G_1 \times G_2$ (in particular, it's a group!).
  (b) Show that if $G_1 \times_\varphi G_2 = G_1 \times G_2$ if and only if both φ_i's are trivial (i.e. send every element to e_H).
  (c) Show that the map π_2 from the first question in this assignment restricts to a surjective homomorpism

$$\pi_2| := \pi_2 \,|_{G_1 \times_\varphi G_2} : G_1 \times_\varphi G_2 \longrightarrow G_2.$$

  (d) Show that $\ker(\pi_2|) \cong \ker(\varphi_1)$
  (This problem is not as bad as it seems as long as one keeps themselves organized! It may be helpful to draw a diagram (directed graph) in the shape of a square with the four functions we are considering as directed edges and the four groups we are considering as the vertices).
  (e) Use Lagrange's theorem and the first homomorphism theorem to show that

$$|G_1 \times_\varphi G_2| = \frac{|G_1|\,|G_2|}{|H|}.$$

  (f) Let

$$\varphi_1 := sgn : S_3 \longrightarrow \mathbb{Z}_2$$

  and

$$\varphi_2 := \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$$
$$a \mod 4 \mapsto a \mod 2$$

[1] For instance if $n = 5$ and $T = \{1, 3\}$, then $(13)(24)$, (245) are in S_T, but (12) is not

(we identify $\{1, -1\}$ with $\mathbb{Z}_2$ in the obvious way) and define $BD_3 := S_3 \times_\varphi \mathbb{Z}_4$. Show that BD_3 is a non abelian group of order 12 which is not isomorphic to D_6 or A_4. (**hint** one way to show that BD_3 is not isomorphic to D_6 is by showing that BD_3 has an element of order 4, but D_6 does not; to show that BD_3 is not isomorphic to A_4 show that BD_3 has an element of order 6 but A_4 does not. Alternatively, one can show that BD_3 has a center while A_4 does not)

(g) Let $H = \langle((1), 2 \mod 4\rangle$ (which happens to be the center of BD_3). Use part (c) and the first homomorphism theorem to show that

$$BD_3/H \cong D_3.$$

(h) In contrast to the previous part, show that BD_3 does not contain a subgroup isomorphic to D_3 (count elements of order 2!).

(8) Suppose that H and K are normal subgroups of G. Show that

(a)

$$HK/H \cong K/H \cap K.$$

(**hint**, define a map from K to HK/H by sending $k \in K$ to $kH \in HK/H$. Then show its surjective and compute its kernel). This is known as the second homomorphism theorem.

(b) If H is contained in K show that $K/H := \{kH : k \in K\}$ is a normal subgroup of G/H and show that

$$G/K \cong (G/H) \,/\, (K/H)\,.$$

This is known as the third homomorphism theorem.