Let $G = S_4$ and $H := \langle a := (1,2,3,4), b := (1,3)\rangle$. H has order 8 and is isomorphic to D_4 (you can see this by labeling the vertices of a square by 1,2,3 and 4 in clockwise order). More explicitly, H is

$$\left\{(1), a = (1234), a^2 = (13)(24), a^3 = (1432), b = (13), ab = (14)(23), a^2b = (24), a^3b = (12)(34)\right\}.$$

Since $(1,2)(1,3)(1,2)^{-1} = (2,3)$ which is not a symmetry of a square (i.e. is not in H), we see that H is not a normal subgroup of G. Never the less, we can still look at the set of $3 = 24/8$ left cosets, G/H.

As calculated in class, the three cosets are $\{H, (1,2)H, (2,3)H|\}$. Now if we label the three cosets by $a := H$, $b := (1,2)H$, and $c := (2,3)H$, then we have that

$$\begin{aligned}
&(1,2)a = (1,2)H = b \\
&(1,2)b = (1,2)(1,2)H = (1)H = H \\
&(1,2)c = (1,2)(2,3)H = (2,3)(2,3)(1,2)(2,3)H = (2,3)(1,3)H = (2,3)H.
\end{aligned}$$

In other words $(1,2)$ takes a to b, and fixes c. In cycle notation $(1,2)$ acts as $(a,b)(c) = (a,b)$. Meanwhile,

$$\begin{aligned}
&(2,3)a = (2,3)H = c \\
&(2,3)b = (2,3)(1,2)H = (1,2)(1,3)H = (1,2)H = b \qquad \text{(since } (1,3) \in H\text{)} \\
&(2,3)c = (2,3)(2,3)H = H = a
\end{aligned}$$

and

$$\begin{aligned}
&(3,4)a = (3,4)(1,2)(3,4)H = (1,2)H = b \\
&(3,4)b = (3,4)(1,2)H = (1,2)(3,4)H = H = a \\
&(3,4)c = (3,4)(2,3)H = (2,3)(2,3)(3,4)(2,3)H = (2,3)(2,4)H = (2,3)H.
\end{aligned}$$

Thus $(2,3)$ acts as (a,c) and $(3,4)$ acts as (a,b). This determines a homorphism, denoted by π,

$$\pi : S_4 = S_{\{1,2,3,4\}} \longrightarrow S_{\{a,b,c\}} \cong S_3.$$

Since every element of S_4 can be written as a product of $(1,2)$, $(2,3)$ and $(3,4)$'s, we can write down the rest of the homomorphism without having to explicitly compute it. (For example one can compute where (13) goes by computing

$$(13) = (12)(23)(12) \mapsto (ab)(ac)(ab) = (bc)$$

)

$$(1) \mapsto (a)(b)(c)$$

$$\begin{aligned}
&(12) \mapsto (ab) \qquad &(23) \mapsto (ac) \qquad &(34) \mapsto (ab) \\
&(13) \mapsto (bc) \qquad &(14) \mapsto (ac) \qquad &(24) \mapsto (bc)
\end{aligned}$$

$$
\begin{aligned}
(123) &\mapsto (acb) & (124) &\mapsto (abc)\\
(134) &\mapsto (acb) & (234) &\mapsto (abc)\\
(132) &\mapsto (abc) & (142) &\mapsto (acb)\\
(143) &\mapsto (abc) & (243) &\mapsto (acb)
\end{aligned}
$$

$$
\begin{aligned}
(1234) &\mapsto (bc) & (1324) &\mapsto (ab) & (1423) &\mapsto (ab)\\
(1243) &\mapsto (ac) & (1342) &\mapsto (ac) & (1432) &\mapsto (bc)
\end{aligned}
$$

and lastly

$$(12)(24) \mapsto (a)(b)(c) \qquad (13)(24) \mapsto (a)(b)(c) \qquad (14)(23) \mapsto (a)(b)(c)$$

Therefore, by inspection, the kernel of this map is (since the identity of the second group is $(a)(b)(c)$)

$$\ker(\pi) = \{(1), (12)(24), (13)(24), (14)(23)\} = V.$$

Furthermore, by inspection, the homomorphism π is also surjective. Therefore, by the 1st homomorphism theorem, we have that $S_4/V \cong S_3$. (notice that indeed, $V < H$).