Bessel Functions

Consider the motion of water waves on a cylindrically shaped object. The motion of the waves can be found by using the classical wave equation (which is a PDE):
(1)

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

and transforming the laplacian into polar coordinates:
(2)

$$\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

where r represents the radial coordinate, θ the angular component, t the time component, and c the constant velocity of the wave disturbance. Solving a partial differential equation with sufficient boundary conditions (called a *boundary value problem*) gives a multivariable relationship using the independent variables (here θ, r, and t) that represents a function of the dependent variable (here u). For example, solving the wave equation in rectangular coordinates will give time varying sine and cosine function. In this talk, I wish to show how solving the wave equation in polar coordinates will have radial dependence on the similar, but slightly more complicated Bessel functions.

A typical technique used to solve many kinds of partial differential equations (P.D.E) is a method known as *separation of variables*. In this technique, the dependent function is assumed to be a simple product of functions of the three independent variables and upon substituting this new relationship and its appropriate derivatives for the dependent function into the original equation in order to get a new relationship for the P.D.E. that has the potential to be written in a form that has one side of the equation dependent on one set of independent variables while the other is dependent on another (hence the terminology *separation of variables)*. To illustrate in the case of our example:
(3)

$$z(r,t,\theta) = \mathrm{R}(r)\Theta(\theta)T(t)\ ,$$

and after making the appropriate substitutions into (2) we get:
(4)

$$\frac{\mathrm{R}''(r)}{\mathrm{R}(r)} + \frac{1}{r}\frac{\mathrm{R}'(r)}{\mathrm{R}(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{1}{c^2}\frac{\mathrm{T}''(t)}{\mathrm{T}(t)}.$$

Notice that each side of (4) is dependent on different variables, which is what was desired (the left having only spatial dependences and the right only time). We must now note that the only way that such a situation may arise is if each side of the equal is equal to a constant. This step in the logic of separation of variables initially may be difficult to fully accept but with thought becomes increasingly obvious when considering what it entails to be an independent variable. This technique when applied to quantum mechanics leads to several famous separation constants such as energy and spin.

Initially, we will call this separation constant *a* (the nature of which will be determined from any initial conditions that may be imposed on the problem):
(5a)

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = a$$

(5b)

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = a$$

Similar logic will apply after multiplying each side of (5a) by an r^2 term and doing subtractions in order to get (now calling the separation constant *b*) a total of three differential equations each of which dependent on only one variable. In addition to (5b) these equations are:
(6a)

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} - ar^2 = b$$

(6b)

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -b.$$

This completes the separation of variables and it is now necessary to solve each differential equation that is now of the regular (non-partial) variety.

It is now important to evaluate the nature of the mutual eigenvalues *a* and *b*. When these particular constants are of the positive variety they will give solutions that are of exponential nature (hyperbolic sines and cosines). However, when noting the physicality of the nature of waves, we realize that requiring our constants to be explicitly negative. To do this, we set $a=-\lambda^2$ and $b=\mu^2$ (the squares are placed in order to remove any ugliness that may later arrive from taking the square roots of terms). From the theory of elementary differential equations, we arrive at the following relations for our functions of time and angle:
(7)

$$T(t) = c_1\cos(\lambda ct) + c_2\sin(\lambda ct)$$

(8)

$$\Theta(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$$

where the terms out front are arbitrary constants.

We are now left with the radial part of the equation (6a), the solutions of which are known as Bessel functions. At this time, I will depart from my example and give a more general approach to these so-called Bessel or Cylindrical functions.

Bessel's equation is an ordinary, linear, homogeneous differential equation given by:
(9)

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

and gives rise to a solution that is a linear combination of a Bessel functions of order *m* (*m* is positive) donated by $J_m(x)$ and a Neumann function of order *m* donated by $Y_m(x)$ the former of which being the simpler and luckily, more applicable solution.

To demonstrate this, we must first consider the *Frobenious method.* From calculus II, we should know that any continuous function might be represented as a McClaurin series of the form:

(10)

$$\mathrm{f(x)} = \sum_{n=0}^{\infty} a_n x^n \,.$$

However, functions that are not continuous at x=0 cannot be represented in such a form. For example, the expansion of the function $\mathrm{f(x)} = \sqrt{x}/\sin x$ can be found to be expressible by the series:

(11)

$$\mathrm{f(x)} = x^{-1/2}(1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + ...)\,.$$

Notice that the series is in the form:

(12)

$$\mathrm{f(x)} = \sum_{n=0}^{\infty} a_n x^{n+s} \,.$$

This provides our motivation to seek solutions of the form of (12) when any sort of discontiniouity is apparent. This is when the coefficient in front of the first term of a differential equation has a value of zero at zero as is the case in the Bessel equation (9).

To summarize the information just claimed, I will now present two theorems without further proof:

Theorem 1: Existence of McClaurin Series Solutions

When the coefficient of the first term of a second order differential equation is solved for unity and the resultant differential equation at x=0 does not have any singular points there exists two linearly independent solutions of the form of (10).

Theorem 2: Frobenius' Theorem

When the coefficient of the first term of a second order differential equation is solved for unity and the resultant differential equation at x=0 has an isolated singularity there exists *at least* one solution of the form of (12).

Notice that in the second theorem that the word *at least* appears. However, from the theorem of elementary differential equations, the most general form of a differential equation must have at least the same amount of solutions as the order of the equation, in the case being considered, two. It turns out that it is important to note three special cases that depend on the nature of the constant *s* that appears in (12).

To determine *s* for a second order homogenous differential equation, one must set (12) equal to the dependent variable and take appropriate derivates then equate all coefficients to zero. After doing so, one will get a quadratic relationship for s (giving two roots s_1 and s_2) known as the indicial equation.

Case 1 If s_1 and s_2 are distinct and do not differ by an integer, then there are two linearly independent solutions of both of the form of (12).

Case 2 If s_1 and s_2 are distinct but do differ by an integer, one must use an important fact from differential equations states that if one solution is found, y_1, then a second *may* be found by:
(13a)

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

(when solving a differential equation of the form *y'' + P(x)y' + Q(x)y=0)*. This works once any $y_1(x)$ is found but is typically saved as a last resort because it can become tedious rather quickly. This equation may be rewritten as (after some calculus and suggestive manipulation):
(13b)

$$y_2(x) = Cy_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+s_2} .$$

Case 3 If s_1 and s_2 are degenerate roots of the indicial equation, then one *must* find a second solution of the for of (13).

The key difference between cases 2 and 3 is the word must. Many times (as we may see in the Bessel case) s_1 and s_2 may differ by an integer but a second linearly independent solution may be found in the form of (12).

Now with some fundamental concepts of the frobenious method at hand, we may now attack (9) in hopes to finally get a view of the storied Bessel functions that thus far seems rather elusive.

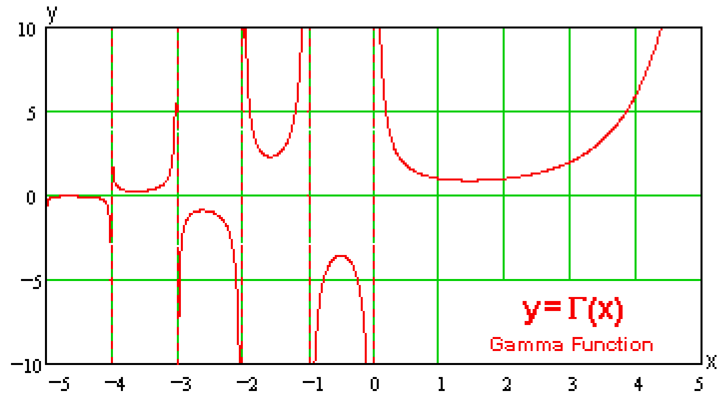
Making the substitution of (13), as well as appropriate derivatives into (9), we can get the indicial equation that has concerned us to be $(s^2-m^2)a_0=0$ which implies that $s=\pm m$. This means that case one applies to us when *2m≠ integer* or *m≠ one half of an integer,* and case three applies when *m=0* and case two applies for any other *m*. Let's initially consider the case of *+m* to get our first solution (which will turn out to be the Bessel functions). After completing the Frobenious method, we get one solution of the form of (13) to be:
(14)

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+m+n)}\left(\frac{x}{2}\right)^{2n+m}$$

Where $\Gamma(1+m+n)$ is known as the gamma function and defined by:
(15)

$$\Gamma(n+1)=\int_0^\infty x^n e^{-x}dx$$

which remarkably (after a repeated integration by parts) turns out to simply be just *n!*. The gamma function makes it possible to define effective non-integer factorials an example being *(1/2)!* $=\sqrt{\pi}/2$. And like the factorial for integers, $\Gamma(n+1)=n\Gamma(n)$. It is also worth noting that the gamma function is unbounded for negative integers as indicated by the figure on the right that shows the values of the gamma function.



To get a second linearly independent function we must first consider the nature of *m.* As discussed previously, if *m* is *not* either an integer or a half integer, we know that it fits into the case 1 category. This means that our general solution (which must include a linear combination of two linearly independent functions) is:

(16)

$$y(x)=c_1J_m(x)+c_2\,J_{-m}(x). \qquad m\neq \text{intergers or half integers}$$

From here, we have two special circumstances where a case 2 scenario may arise.

The first circumstance is where *m* is a half and odd integer. Remarkably, not only does (16) also hold for this case, but the solutions reduce to linear combinations of sine and cosine functions. This is an example of a case where the difference in the s values is an integer, but we still get two linearly independent functions of the form of (13). The case where *m* is a half integer is so special in fact that it is given its own name: spherical Bessel functions. Spherical Bessel functions arise in many important cases such as wave equation of the universe's most common element, Hydrogen. Several spherical Bessel functions are listed below[1].

$$\mathrm{J}_{-1/2}(x)=\mathrm{Y}_{1/2}(x)=-\sqrt{\frac{2}{\pi x}}\cos(x) \qquad \mathrm{J}_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin(x)$$

$$\mathrm{J}_{-3/2}(x)=\mathrm{Y}_{3/2}(x)=-\sqrt{\frac{2x}{\pi}}\left(\frac{\cos(x)}{x^2}-\frac{\sin(x)}{x}\right) \quad \mathrm{J}_{3/2}(x)=-\sqrt{\frac{2x}{\pi}}\left(\frac{\sin(x)}{x^2}-\frac{\cos(x)}{x}\right)$$

A second circumstance that fits into case 2 is the case where *m* is a full integer. This is another important case, and when we return to our example, we see that it arises there in particular. Here, we are not so lucky to get two independent solutions as I will now illustrate.

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[1] Often, spherical Bessel functions are listed by multiplying the ones above by a factor of $\sqrt{\frac{2x}{\pi}}$.

We are looking for the relationship between J_m and J_{-m} where m=1,2,3… (recall that it was originally stated when the problem began that *m* is not negative). Putting this into the series representation of the Bessel function (14), we see that:

$$\mathrm{J}_{-m}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-m+n)}\left(\frac{x}{2}\right)^{2n-m}$$

However, we had previously noted that the gamma function was not bounded for negative integers, so each term (n=0,1…m) must vanish (dividing by infinity), and we restart our series at *n=m*.

$$\mathrm{J}_{-m}(x)=\sum_{n=m}^{\infty}\frac{(-1)^n}{n!\Gamma(1-m+n)}\left(\frac{x}{2}\right)^{2n-m}$$

We wish to start our series at zero again, so we define a new variable as follows: $z\equiv m-n$. This does the job, and we get:

$$\mathrm{J}_{-m}(x)=\sum_{z=0}^{\infty}\frac{(-1)^z}{(z+m)!(z)!}\left(\frac{x}{2}\right)^{2z+m}$$

$$\mathrm{J}_{-m}(x)=(-1)^m\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+m)!(n)!}\left(\frac{x}{2}\right)^{2n+m}=(-1)^m\,\mathrm{J}_{-m}(x)$$

Clearly, we do not get independent solutions here. Thus, we must resort to either form of (13) to give us a second independent solution when *m* is an integer. And of course, the same must be done when *m=0* which is the necessary condition to fall into case three (which always requires the use of (13)). With the fact that in the later case there will be a recurrence relationship defined, only the first such solution is (the case where m=0):

(17)

$$\mathrm{Y}_0(x)=\frac{2}{\pi}\mathrm{J}_0(x)\left[\gamma+\ln\frac{x}{2}\right]-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)\left(\frac{x}{2}\right)^{2n},$$

Where γ is the word's third most prominent mathematical constant, Euler's constant given by 0.57721566…(it has yet to be determined whether this number is rational or not, so if anybody is interested, there's a good time install). To determine the other orders of $\mathrm{Y}_m(x)$ and to see where the constants come from, one can use a defined relationship for $\mathrm{Y}_m(x)$:

(17b)

$$\mathrm{Y}_m(x)=\lim_{p\to m}\frac{\cos(p\pi)J_p-J_{-p}}{\sin(p\pi)}.$$

Notice that one must use L'hopital's rule when p is an integer. Interestingly enough, Euler's constant in (17a) comes out when taking the derivative of the gamma function.

Now that two linearly independent solutions have been presented that will cover any order Bessel function, we can explore some of the properties of these mathematical functions. From the form of (14), we can conclude that a Bessel function of an odd integer order is considered an odd function while a Bessel function of an even integer order is an even function. Also, it useful to derive the following relationship between

Bessel functions of different orders and their respective derivatives. This is typically done by letting

$$J_m(x) = x^m f(x)$$

so,

$$\left(\frac{J_m(x)}{x^m}\right)' = f'(x) = \sum_{n=1}^{\infty}(-1)^n \frac{1}{(n-1)!\Gamma(n+m+1)2^{2n+m}} x^{2n-1}$$

And after bringing the sum back to start at one by replacing n by n+1 it is easy to see that:

(18)

$$\frac{d}{dx}\left(\frac{J_m(x)}{x^m}\right) = -\frac{J_{m+1}(x)}{x^m}.$$

Another such identity can be found by employing similar techniques and reads

(19)

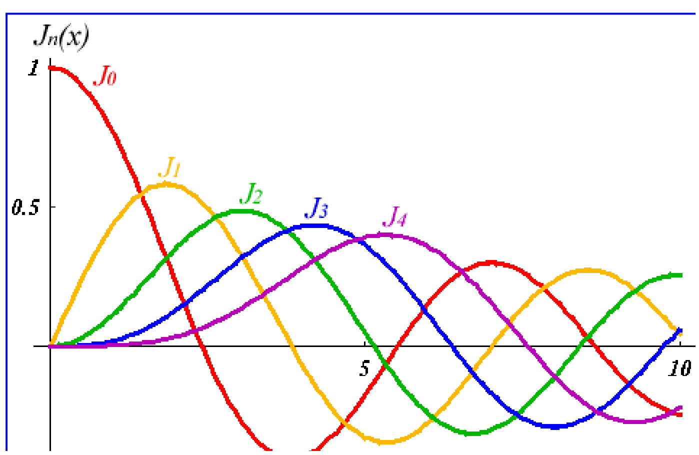
$$\frac{d}{dx}\left(x^m J_m(x)\right) = x^m J_{m-1}(x).$$

These two relations are extremely useful when confronted with the evaluation of integrals and derivatives. Also, they also apply to Neumann functions, and they establish a means in which we can establish the values of the other such functions from (17).

One special case of (18) is the derivative of $J_0(x)$. Here, since $m=0$, we have

$$\frac{d}{dx}J_0(x) = -J_1(x).$$

Notice, for the second time (the first time being the evenness of $J_0(x)$ and the oddness of $J_1(x)$) we have drawn a comparison to the trigonometric functions. It is worth exploring these similarities further. Pictured on the right are the Bessel functions of orders 0 through 4. Clearly, like trigonometric functions, they have infinitely many zeros and it can be show that spacing between these zeros approach π (however, the zeros of the function are not spaced by the same amounts as they are in trigonometric functions). Also, like the cosine function, $J_0(0)=1$ while like the sine function $J_1(0)=0$. Inspection of the image on the right shows that the zeros of $J_0(x)$ correspond to the mins and maxs of $J_1(x)$.



We are finally ready to return to our initial wave equation example. Where we last left it, we were left with (6a). After that big discussion on Bessel functions, I will now show how this equation has Hermite polynomials for solutions. Just kidding, we will show how Bessel functions arise from the motion of waves on a cylinder. Doing this is just a simple matter of letting r=xλ and multiplying through by the *R(r)* term. This reduces (6a) to:

(20)

$$x^2 \mathrm{R}''(x) + x\mathrm{R}'(x) + (x^2 - \mu^2) = 0$$

Which is the Bessel equation that we have been wrestling with for some time now, and it has solutions that read:

$$R(x) = C\,\mathrm{J}_n(x) + D\,\mathrm{Y}_n(x)$$

or,

(21)

$$R(\lambda r) = C\,\mathrm{J}_n(\lambda r) + D\,\mathrm{Y}_n(\lambda r)\,.$$

Up till this point, physical parameters that will affect the system (the so-called boundary conditions) have not been presented. In order to truly be a boundary value problem, there most be a way that an exact solution can be generated. For this reason, there will be five such conditions discussed below.

(i) At no point may the function z(r,θ,t) be greater then some set value let's call it M. This ensures a trivial implication that we are dealing with a bound function at all times and locations. This implies that our second arbitrary coefficient, D, in front of the Bessel function of the second kind must be set equal to zero to ensure this (other wise, the solution would not be bound at r=0 because Y_n(r) is not less then our M here).

(ii) To eliminate the need to have to end up doing a Fourier and Bessel expansion, the provision will be imposed that axial symmetry exists. This would imply that our angular function would initially have to equal a constant. And since the initial functions are arbitrary, the only way to ensure this is imposed is to limit *m* to be zero and permanently making our angular component yield a constant value. This is doubly nice because it will limit our Bessel functions to just that of the zero order variety.

(iii) The condition of an initial condition will be determined by shape alone with no initial velocity function. This means that the first derivative in respect to the time component (7) must equal zero at t=0. Taking this and replacing *t* by zero implies that the coefficient in front of the sine term must vanish.

(iv) The initial displacement function will be given by *z(r,0)=W(r)*.

(v) The object on which the wave motion occurs has a length *a* and all motion must vanish at that boundary *(i.e. at r=a,* $J_0(\lambda r)=0$*)*. What this implies is:

$$C\,\mathrm{J}_0(\lambda r) = 0$$

And to avoid the trivial solution, we do not wish to set C=0 but to let $\lambda c = \alpha_j$ where α_j is the j[th] zero the Bessel function. This now allows to solve for λ in terms of the zeros of the Bessel function and the length of the wave medium present. As you can see from the graphs of the Bessel function, there are infinitely many of these roots, and as the theory of differential equations requires, the most general solution is a linear combination of every possible solutions.

We are finally ready to bring our z(r,t) to life. Recall, from back when we had said that z(r,t) a product of the two independent functions that we have been dealing with for quite some time now. Substituting values for our three functions we arrive at the solution:
(21)

$$z(r,t) = \sum_{j=0}^{\infty} A_j \, \mathrm{J}_0(\frac{\alpha_j r}{c}) \cos(\frac{\alpha_j t}{c})$$

where A_j is the product of all constants that still remain. As of this point, we still have not gotten the most general solution that we seek after. To do this we must have no remaining undetermined constants remaining in our function. Luckily, however, we still have yet to impose boundary condition (iv). In order to take advantage of this condition, we must set or time parameter to zero and set the resultant function equal to the W(r):
(22)

$$\mathrm{W(r)} = z(r,0) = \sum_{j=0}^{\infty} A_j \, \mathrm{J}_0(\frac{\alpha_j r}{c}) .$$

At first appearance, this may not appear to have helped us, but we are going to be able to use a property of Bessel functions in which we will be able to extract values for A_j.

We must now again pause to see what it entails to find the projection of a function onto an orthogonal basis.

Definition 1:
Orthogonal – A function is orthogonal if a defined inner product vanishes between two unlike components of a particular inner product space (an inner product between a function Ψ_a and Ψ_b shall be depicted mathematically by $\langle \Psi_a | \Psi_b \rangle$).

The simplest example of an inner product is a dot product. One may be able to recall from physics courses as well as linear algebra courses the standard basis of $\Re^3$, $\hat{i}, \hat{j}$ and $\hat{\mathrm{k}}$.. A dot product is defined as the summed product of like components of that particular vector. Since the standard basis has no like components with each other, it is easy to see that it defines an orthogonal set. Also, the standard basis is also a set of unit vectors.

Definition 2:
Normal Vector/Unit Vector - A component of an inner product space is normalized if the inner product with itself is equal to unity.

Definition 3:
Orthonormal - A component of an inner product space is orthonormal if it is both orthogonal and of unit length, mathematically:
(22)

$$\langle \Psi_a | \Psi_b \rangle = \boldsymbol{\delta}_{a,b} \equiv \begin{cases} 0 \text{ if a} \neq \text{b} \\ 1 \text{ if a} = \text{b} \end{cases},$$

where $\boldsymbol{\delta}_{a,b}$ is the so-called Kronecker Delta symbol and is defined as above.
When a situation comes about where (22) is applicable, one can use it to find the projection of different components on the particular orthonormal basis of an inner product space. Let your inner product space be defined for the set of all vectors in $\Re^3$ with the inner product defined to be the dot product and the orthonormal basis to be the standard basis vectors mentioned earlier. Let us call any arbitrary vector with three dimensions:
(23)

$$\text{V} = \sum_{j=0}^{3} A_j e_j$$

where e_j is the j[th] standard basis vector (either $\hat{i}, \hat{j}$ or $\hat{\text{k}}$). A more concrete example would be V=$3\hat{i} + 5\hat{j} + \hat{\text{k}}$. The situation of being able to find the A_j resembles our current problem with our Bessel functions. Once an orthonormal basis has been found, the problem is rather simple as in the case of the above example (23):

$$\langle \text{V} | e_i \rangle = A_j \, \boldsymbol{\delta}_{i,j}$$

(24)

$$\langle \text{V} | e_i \rangle = A_i$$

This is a very important formula and relies on the ability to be able to find an a set that is orthonormal (at least to some constant). In the case of a non-normalized orthogonal basis:
(25)

$$\frac{\langle \text{V} | \Psi_a \rangle}{\langle \Psi_a | \Psi_a \rangle} = A_j \boldsymbol{\delta}_{i,j}$$

This is the standard projection equation encountered in linear algebra of a vector V onto a basis vector Ψ_b .

Theorem 1: Orthogonality of the Bessel Functions
Bessel functions of different eigenvalues are orthogonal on the inner product space with an inner product defined to be:

$$\langle y_a \,|\, y_b \rangle = \int_0^c x y_a y_b dx \ .$$

Proof:

To do this we need to go back to the original Bessel differential equation (20) with the reinstallation of the eigenvalue λ.

$$x^2 \, R''(x) + x \, R'(x) + (\lambda^2 \, x^2 - \mu^2) = 0$$

This equation has as its solution y_1= $J_m(\lambda x)$. Also, we will consider another such Bessel differential equation with a different eigenvalue say y_2= $J_m(qx)$ and multiply the former differential equation by y_2 and the latter by y_1. After subtracting each of these equations from each other and some manipulation, we find

(26

$$\frac{d}{dx}\{x[y_2 y_1' - y_1 y_2']\} = (q^2 - \lambda^2) x y_1 y_2 \, .$$

Notice that the term inside of the derivative vanishes after integrating and evaluating and the limits specified in the boundary conditions (the function must vanish at the boundary or y(c)=0)! This leads:

(27

$$0 = \{x[y(c)_2 \, y'(c)_1 - y(c)_1 \, y'(c)_2]\} = (q^2 - \lambda^2) \int_0^c x y_1 y_2 dx \, .$$

It is apparent from (27) that either $(q^2 - \lambda^2)$ must equal 0 or the integrand does. This is was the motivation to define the inner product the way that it was done and leads to:

(28)

$$\langle J(\lambda x) \,|\, J(qx) \rangle = \int_0^c J(\lambda x) J(qx) x dx = \langle J(\lambda x) \,|\, J(\lambda x) \rangle \delta_{\lambda,q}$$

And this completes the proof.

One final question remains: what is the value of the normalizing constant $\langle y_\lambda \,|\, y_q \rangle$? To get this, a limit of q-> λ needs to be taken and use of (18) and (19) gives us:

$$\langle J(\lambda x) \,|\, J(\lambda x) \rangle = \int_0^c x J(\lambda x) J(\lambda x) dx = \frac{2c}{J_{n+1}^2 (\alpha_\lambda / c)} \, .$$

Now that we have a normalizing constant and validation that we are indeed dealing with an orthogonal basis, we can finally use (25) to pick out our weighting factor A_j in (22) and substitute it into (21) to get our particular solution to the problem started at the beginning of the paper.