Coxeter Group Actions on CAT(0) spaces

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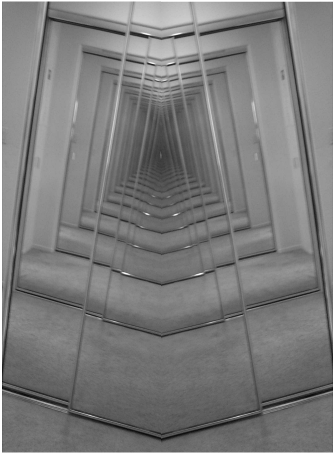
Outline

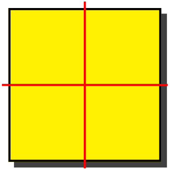


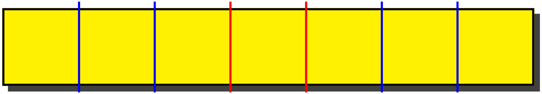
Right-Angled Coxeter Group Def

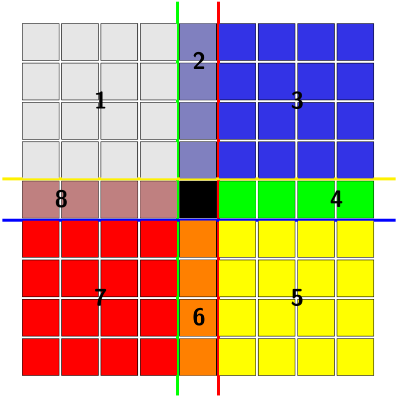
Definition

Let Γ be a (simplicial) graph. A Right-Angled Coxeter Group associated to Γ is a group generated by a set of elements in bijection with the vertices of the graph with the requirements that

1. Every generator has order 2.
2. The product of two generators has order two if and only the corresponding vertices are connected by an edge in Γ.



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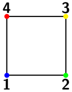
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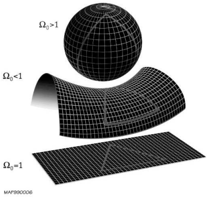
A Rough Definition of a CAT(0) space

Definition

A CAT(0) space is a metric space where

1. Any two points are connected by a path of length equal to the distance between the two points.
2. The sum of the angles (!) of any triangle add up to no more than π.



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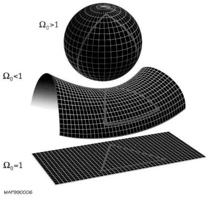

About the Davis Complex

A Pseudo-Definition

The Davis complex is a $\mathrm{CAT}(0)$ space with admits a geometric action of a right-angled Coxeter group with the following properties

1. The fixed point set of any generator separates the Davis complex into exactly two components.
2. The action of a generator swaps these two components.
3. The action of the link (set of generators at distance 1 in the graph) preserves these two components.
4. The fixed point set of every generator in the anti-link (set of generators of distance greater than 1) of a given generator lies in the *same* component of the complement of the fixed point set of the given generator.

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The Definition

Throughout the talk, I keep the underlying assumption that W acts geometrically on X and X is a $\mathrm{CAT}(0)$ space.

Definition

A strict fundamental domain for $W \curvearrowright X$ is a set $K \subset X$ which satisfies the following:

- K is a closed subset of X.
- Every orbit of $W \curvearrowright X$ meets K in exactly one point.

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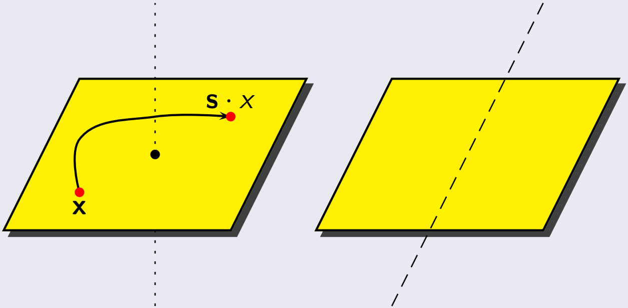
Examples

W/out Strict Fundamental Domain, W/



Properties of Strict Fundamental Domains

Why we care about strict fundamental domains:

1. The retraction $X \to K$ given by $x \mapsto W{\cdot}x \cap K$ (*i.e.* x gets sent to the unique point in its orbit that meets K) induces an identification of X/W with K.
2. Can build X up from K and W via $(W \times K)/\sim$ where $\sim$ is some naturally defined equivalence relation.
3. A strict fundamental domain determines a presentation of the group.

First point implies that W cannot be torsion free.

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In the remainder of the slides (W, S) is a right-angled Coxeter system, meaning that W is a right-angled Coxeter group as defined earlier and S is the generating set (in bijection with the set of vertices of the defining graph).

The Main Theorem

Definition

Suppose $T \subset S$ generates a finite group (i.e. is spherical). We denote w_T as the unique maximal element in W_T.

Theorem

Suppose (W, S) acts geometrically on a $\mathrm{CAT}(0)$ space. Then there exists necessary and sufficient conditions based entirely on the separation properties of the w_T for $T \subset S$ for the existence of a strict fundamental domain. Moreover the construction of the strict fundamental domain which gives us a presentation (W, S') tells us exactly which automorphisms to use to change S into S'.

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Applications

1. Applying the main theorem to the Davis complex gives a geometric proof of rigidity of right-angled Coxeter groups (proven in the Radcliffe, 1999).
2. Simultaneously gives geometric proofs of many well known facts about $\mathrm{Aut}(W)$ (Tits, 1988 as well as others).
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The Special Case

Due to ease, I will start with Universal Coxeter systems:

Definition

A Universal Coxeter System is simply a pair (W, S) where W is a group with presentation

$$\langle s_1, ..., s_n \mid s_1^2, .., s_n^2 \rangle$$

and S is the generating set $\{s_1, ...s_n\}$.

This is the same thing as

1. Coxeter system without relations
2. Free product of $\mathbb{Z}_2$'s with obvious generating set.

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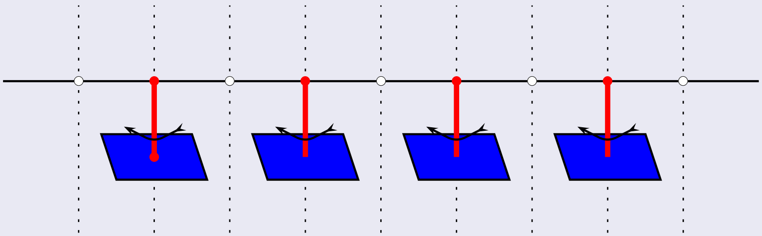
In the case of Universal Coxeter systems,

Definition

A universal Coxeter system acts by generalized reflections if for every $s \in S$ and every $x \in X$ every path from x to $s \cdot x$ meets the fixed point set of s.

Not a Generalized Reflection System



The Main Theorem Restated in Terms of UCS

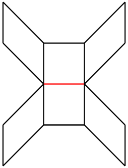
Theorem

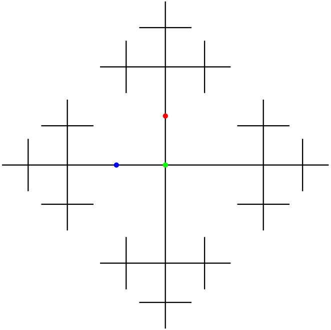
A universal Coxeter system (W, S) has a strict fundamental domain if and only if it acts by generalized reflections.

Example

Every action on a tree is by generalized reflections. However here the theorem stated above is just a special case of Bass-Serre theory.

A couple of examples





Automorphisms of Universal Coxeter Groups

Fact

Every automorphism of a Universal Coxeter system is a product of partial conjugations and graph symmetries.

Definitions

1. Fix a generator $s \in S$ and a subset $T \subset S \setminus \{s\}$. A partial conjugation is the automorphism induced by conjugating every element of T by s and leaving every other generator alone.
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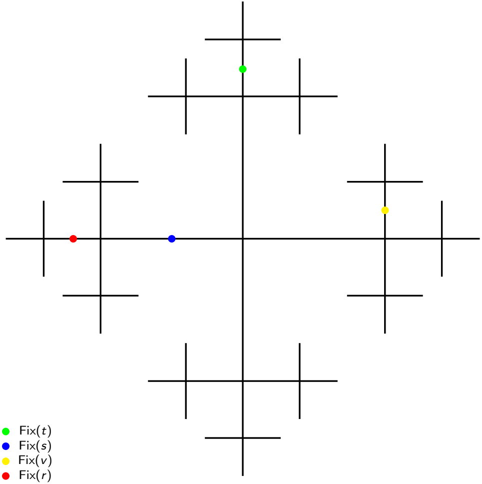
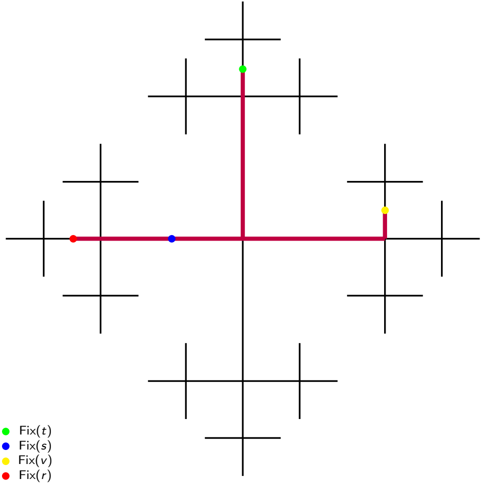
Action on a Tree



- Fix(t)
- Fix(s)
- Fix(v)
- Fix(r)

Why Isn't the Purple a Strict Fundamental Domain?



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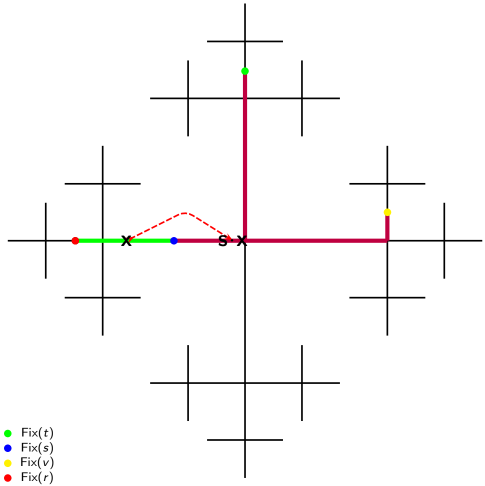
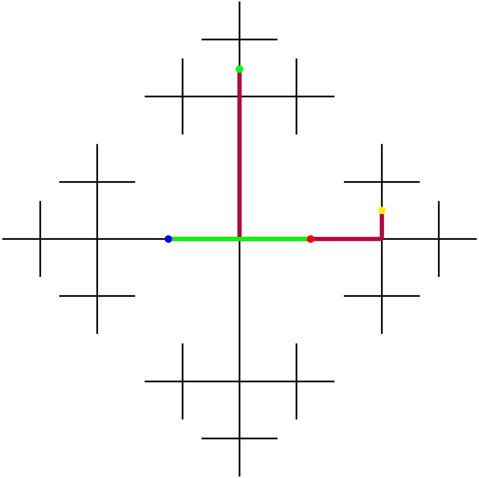
$W \cdot x$ Meets Region in 2 Points



Apply Partial Conjugation by Conjugating r by s



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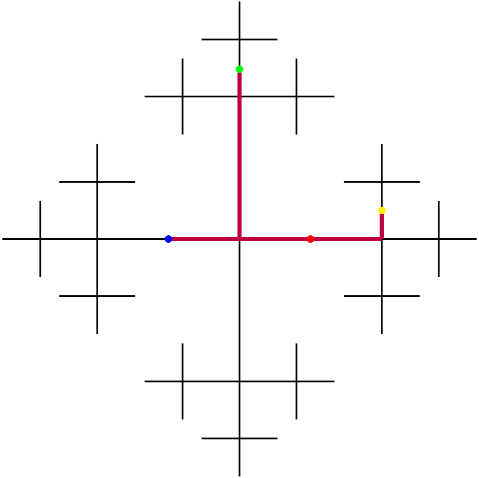

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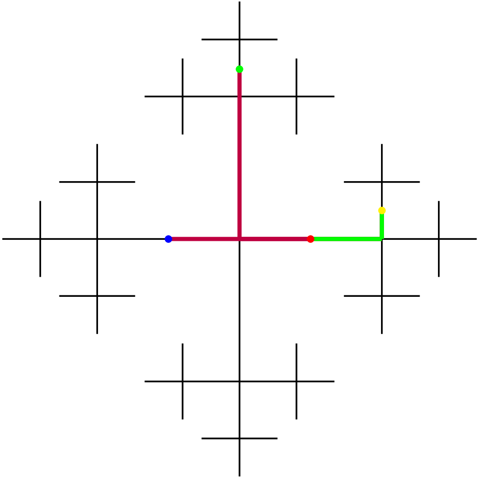

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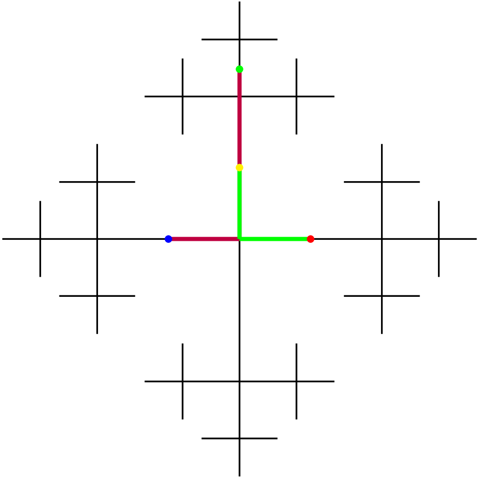

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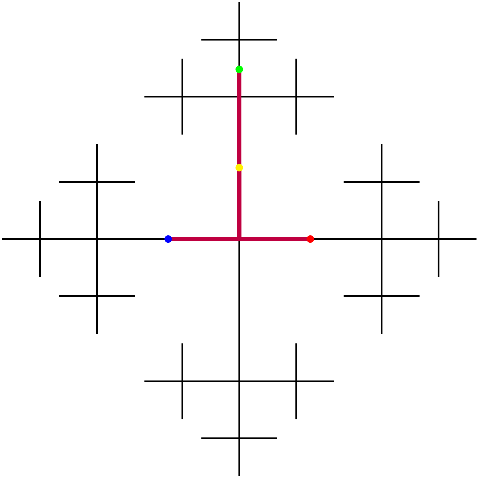

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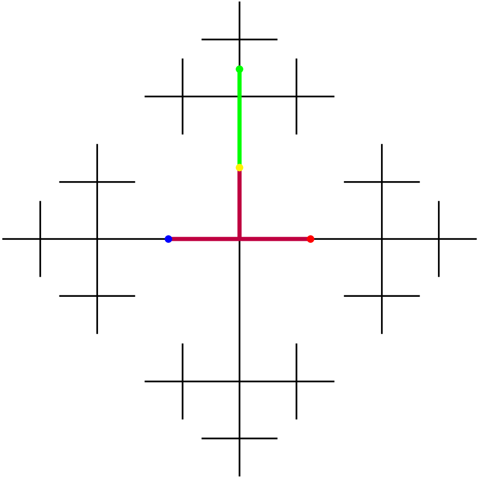

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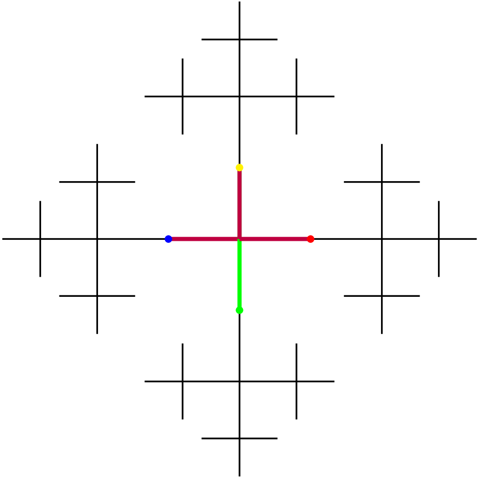

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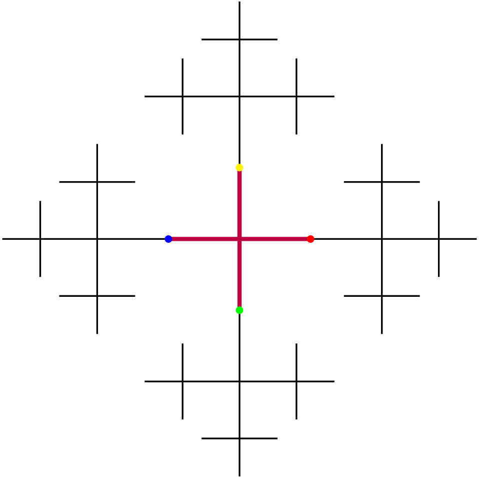
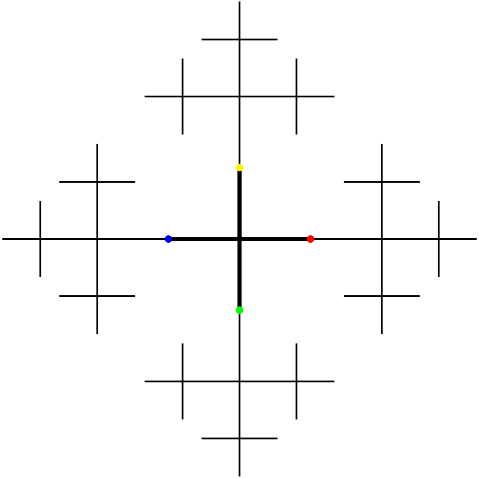


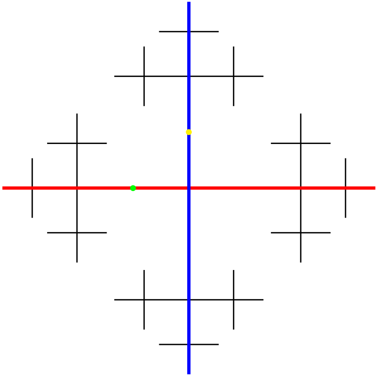
The Black Region is a SFD



A pathological example in Non-Universal Case

Some Pathologies

1. Every x in this space is in the fixed point set of some reflection.
2. Removing the fixed point set of either the blue or the red generator separates space into an infinite number of components.



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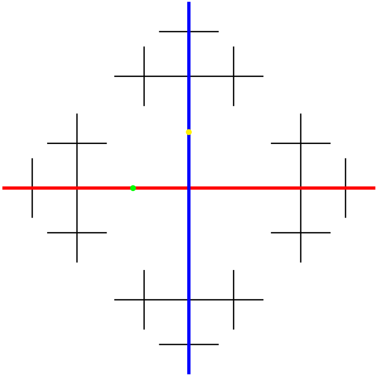

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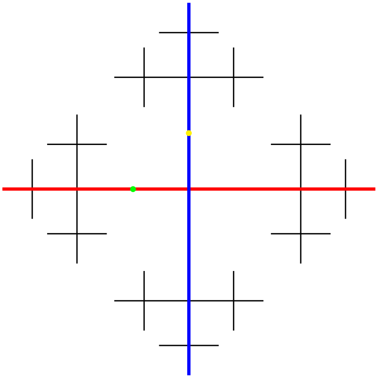
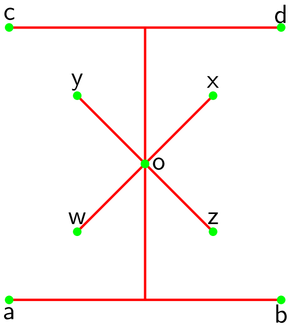


Commutation Is Harder

In the general case, commutativity of generators makes things significantly more complicated.

$s_1 \cdot$



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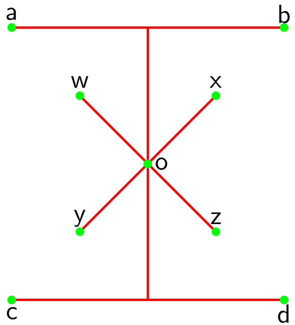

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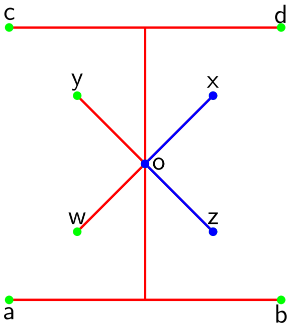

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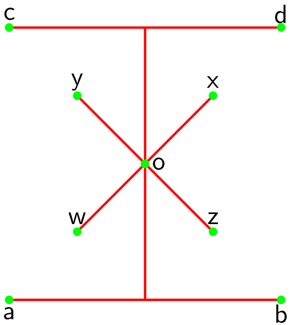
s_2:



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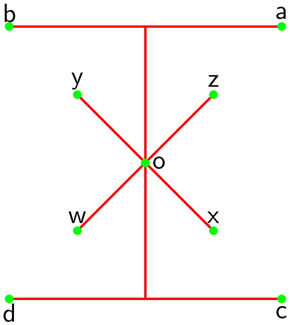

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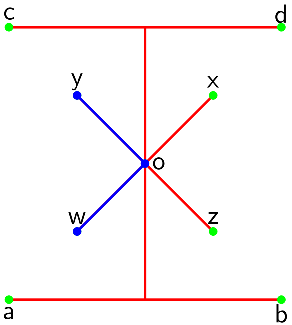

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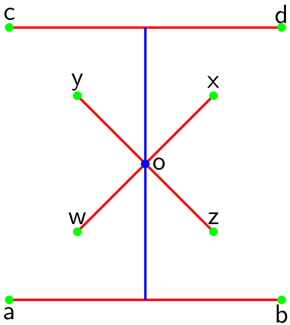

The SFD can be taken to be

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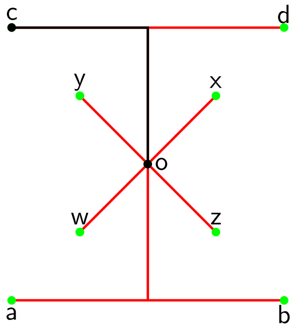
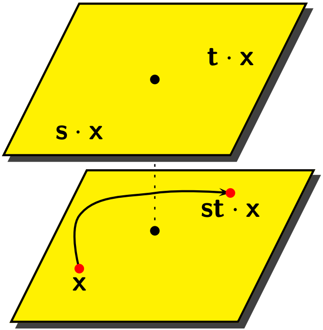


A similar example without a SFD



(The two squares are glued together along the dotted line. Also glue in “wings” as in the previous example)

A first approximation to the main theorem

Theorem

Suppose for every s-spherical T and C a component of $X \setminus X^s$ we have that if $w_T \cdot C = C$, then every path from $x \in C$ to $w_T \cdot x$ meets X^{w_T}. Then there exists a strict fundamental for the action of W.

Remark

This is only a first approximation because this condition is only sufficient.

Step 1

Turning

As indicated in an earlier slide, maybe for commuting s and t we have $X^s \cap X^t \neq X^{st}$. To fix this problem I apply the turning procedure which replaces the original action with a new action with the properties that

1. The orbit of every point in X is the same under the new action.
2. The new action no longer has the original problem.

Step 1

Remark

In the case of the Davis complex (where if a fixed point set separates the space, then it separates it into two components) this turning process corresponds directly to a specific kind of automorphism known as a transvection.

Step 2

Folding

It is possible that for a single generator s that for a component C of $X \setminus X^s$ that both C and $s \cdot C$ contain the fixed point sets of generators which do no commute with s. To fix this problem I apply the folding procedure. Just like in the universal case, this corresponds to applying automorphisms to the original right-angled Coxeter group.

Remark

This is the same problem that arose in the universal case and (after a lot of work!) it has the same solution.

Step 3

Bounded Components

Even after turning and folding it may be the case that for a given generator $s \in S$ that the orbits of components of s which contain the fixed point set of a generator not commuting with s does not exhaust all of the components of s (for example if s is central, then this is the case for all components of s). If this is the case, the components lie a bounded distance from X^s. These so called bounded components must then be carefully glued onto a potential strict fundamental domain in order to facilitate an induction argument.

Step 4

Complete the Proof

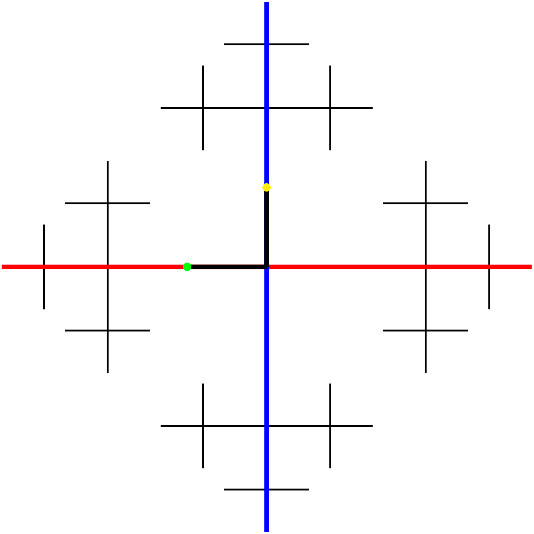
For non-commuting $s, t \in S$, let T^y_{st} be the unique component of s which contains the fixed point set of t and $\mathrm{T}^n_{st} := s \cdot \mathrm{T}^y_{st}$.
For a given $s \in S$ the third step shows that there are only two kinds of components that matter, namely orbits (under the link of s) of T^y_{st} and T^n_{st} for various t in the anti-link of s (denoted $\mathrm{kl}(s)$). Moreover unwinding the first and second step guarantees that these two orbits are disjoint. The proof finishes by applying the identical arguments that one would apply in the case that there were only two components of every generator.

Step 4

Remark

In the case that we don't have to worry about bounded components (which is less often then we'd like) the strict fundamental domain of the action is

$$\widetilde{\mathrm{T}^y} := \bigcap_{s \in S} \bigcup_{t \in \mathrm{kl}(s)} \left(\mathrm{T}^y_{st} \cup X^s\right)$$



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