Mark O'Brien - Research Statement

My research is in geometric group theory, with a particular focus on Coxeter groups and their actions on CAT(0) *spaces.*

Typically, geometric group theory studies the relationship between a group and a space in which the group acts properly (discrete orbits and finite stabilizers) by isometries with compact quotient. These actions play such a central role in the field that they are known as *geometric actions*.

Covering space theory shuns actions with non-trivial fixed point sets. Indeed, the fundamental group of the quotient can be identified with the original group only in the case of a free action. Moreover, when the space is contractible and finite dimensional, the group must be torsion free. However, Coxeter groups have a lot of torsion yet when acting on a contractible space something quite different but equally beautiful occurs - namely, a *strict fundamental domain*. A strict fundamental domain is a fundamental domain for the action that is contractible and can be identified with the quotient, a situation which is diametrically opposite from the covering space action. Moreover, the orbit of the fundamental domain under the Coxeter group tessellates the space. Finally, this strict fundamental domain when coupled with suitable stabilizer information completely determines both the group presentation and the original space.

Thesis Summary

Finding a strict fundamental domain for an action is a compact and elegant way in which one can understand a wide range of properties of both a space and the acting group. Unfortunately, not every geometric action of a right-angled Coxeter group has a strict fundamental domain. For instance, a 180° rotation of a square does not have a strict fundamental domain. Moreover, if this square action equivariantly embeds into a larger space we can not have a strict fundamental domain. A major part of my thesis involves coming up with an easy-to-check definition of a reflection which is equivalent to the existence of a strict fundamental domain. In the case where non-trivial products of commuting generators are reasonably behaved the only requirement is that every generator acts freely on the components of the complement of its fixed point set. If these products are not reasonably behaved there is a slightly more technical definition.

In this paragraph I state precisely the hypothesis that are used in my thesis. This paragraph is separated because it is more technical then the informal statements that were made in the previous paragraph. I say that a pair (W, S, X) is a right-angled Coxeter pair if

- W is a right-angled Coxeter group which acts geometrically on a CAT(0) space X.
- If w consists of pairwise commuting letters and stabilizes a component of $X \setminus \mathrm{Fix}(s)$ (s a letter in w), then for every x in that component, every path from x to $w \cdot x$ meets $\mathrm{Fix}(w)$, the fixed point set of w.

My main theorem gives an explicit construction which takes a region of the set X associated with the essentially arbitrary generating set S and transforms it into a strict fundamental domain for the action. In particular this shows that whenever we have a right-angled Coxeter pair there *exists* a strict fundamental domain. In the previous paragraph I stated that if products of commuting elements are "reasonably behaved" the condition for being a right-angled Coxeter pair is much easier to check. I say that products are reasonably behaved if the condition for being a right-angled Coxeter pair is vacuous, that is if w does not fix any component of $X \setminus \mathrm{Fix}(s_1)$.

One consequence of my research is that most spaces with geodesic extendibility have the property that every geometric action by a right-angled Coxeter group has a strict fundamental domain; one only needs that the space is not formed by gluing together spaces that do not have geodesic extendibility together in order to get a space which does. For instance, if we equivariantly glue together two planes along a line segment we get a geodesically extendible space which admits a right-angled Coxeter group action which does not have a strict fundamental domain.

Even though I never assume that the action separates the space into exactly two components, I naturally obtain results which are necessary and sufficient for this to happen. In the literature the equivalent condition is referred to as *strong rigidity*. Moreover, like many other results from my thesis this is obtained as a consequence of the proof (more precisely it is a consequence of what is *not* needed in the proof) and not even the slightest detour is needed.

My construction breaks down the question of which right-angled Coxeter groups can act on a particular space to a purely combinatorial question. For instance, in the case that the original space is a tree my research is equivalent to Bass-Serre theory.

Future Research

The most obvious question that one can ask is whether or not my research can be generalized to all Coxeter groups. For one thing, since general Coxeter groups are not rigid it is impossible to be able to naturally form a strict fundamental domain from the original configuration of fixed point sets in the same way that I have done in my thesis. Even for rigid Coxeter groups which are not right-angled it is difficult to understand what it means for higher products to be well behaved in the sense that I mentioned in the previous section. What makes the right-angled case easier is the fact that all torsion elements have order two. Even more specifically the only Coxeter groups with the property that the centralizer of each generator is special are right-angled. Indeed, much of my proof depends heavily on this fact.

While I have conditions that are easy to check for a particular *space*, I would like to be able to give conditions for the *group* which guarantee that any geometric action (on a space with geodesic extendibility) has a strict fundamental domain. This seems to be the case quite often and the only examples I can find at this point involve certain amalgamated products over groups which are much smaller than the rest of the group. Based on the corresponding property for spaces discussed in the previous section this sort of result makes sense.

One aspect of my research that I find particulary interesting is how the study of the automorphism group of a right-angled Coxeter group came about quite naturally. Thus even though much is known about the structure of the automorphism group and the rigidity of right-angled Coxeter groups, the proofs in the literature are all combinatorial. My proof gives some geometric insight into these two properties for right-angled Coxeter groups. Indeed, even though I have never studied these directly, I have come to understand much about it. For instance, it is known that modulo permuting the defining generators, the automorphism group of a right-angled Coxeter group has exactly two types of generators. These two types of generators turn out to be precisely what I use in order to pick out the strict fundamental domain.