Right-Angled Coxeter Groups and CAT(0) Spaces

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Dedicated to the memory of my father. I think that he would have been very proud.

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Contents

Introduction

As the name indicates, geometric group theory studies the relationship between geometry and group theory. More precisely, if a group G acts geometrically (see Definition 1.3.6) on a geodesic metric space X, how do the geometric properties of X determine group theoretical properties of G, and conversely, how do the group theoretical properties of G determine the geometric properties of X?

In the first instance there are are many examples of concise statements. For instance, if G acts geometrically on $\mathbb{R}^n$ (resp. a tree), then G contains $\mathbb{Z}^n$ (resp. a free group) of finite index. This shows among other things that $\mathbb{Z}^n$ admits no geometric action on a tree while F_2 (the free group of rank two) admits no geometric action on the plane.

The converse question is not quite as satisfactory, especially on the local scale. For instance, if one considers the Cayley graph of $\mathbb{Z}^2$ and the space $\mathbb{R}^2$ these two spaces from a local perspective have nothing in common. This is despite the fact that both spaces admit geometric actions of $\mathbb{Z}^2$. However, if one steps back, these two spaces look identical; just ask anybody who has walked into a screen door.

However for many purposes this “far away” notion is not satisfactory. For instance in [CK00] the authors show that there are two CAT(0) spaces admitting geometric actions of the same group which have different visual boundaries (see [BH99] for definitions). This is despite the fact that the visual boundary is intuitively a “far away” concept.

One way that one can try to reconcile with this disheartening fact is by looking at the fundamental domains of the groups actions of the two spaces and noticing their difference (one is made of three squares glued together while the other is of two squares glued to a slanted parallelogram). This sort of difference should not arise when studying Coxeter groups simply because the notion of angle is inherently part of the Coxeter group definition.

Unfortunately, just because a group has a given presentation does not guarantee that the actions fundamental domain in any way reflects this given presentation. The geometry determines the presentation, not the other way around. Thus given the original presentation of the group and passing to the new presentation determined by a certain canonical fundamental domain ought to be as difficult as finding the required Tietze transformations which go between the two said presentations and even more difficult than solving the word problem in the automorphism group of G.

In the above paragraph the reader might have asked him or herself one or all of the following questions:

1. What would be a canonical fundamental domain?
2. Does such a fundamental domain exist?
3. When can we use the geometry to actually find a way to go between the two presentations?

To answer the first question, a canonical fundamental domain ought to be the smallest possible fundamental domain. That is a fundamental domain which meets every orbit exactly once. If one moreover requires that this fundamental domain is closed, then it will allow one to rebuild the original space simply from it and stabilizer information. Such a fundamental domain is known as a *strict fundamental domain*. It has the special property that it can actually be identified with the quotient X/G (see 2.1 for more on strict fundamental domains in general).

Alas, for a given group action to have a strict fundamental domain is, well, a rather strict condition. Indeed, assuming X is contractible (as it always will be in this document), then the group G must be generated by torsion (the set of elements of G whose fixed point sets meet the strict fundamental domain). Moreover even for the simplest group generated by torsion, $\mathbb{Z}_2$, the action of it on the usual square in the plane by rotation by 180 degrees does not have a strict fundamental domain (having a strict fundamental domain ought to, in some sense, correspond to the group G acting by reflections).

The point of this thesis is to give necessary and sufficient conditions for the existence of a strict fundamental domain for a geometric action of a right-angled Coxeter group on a CAT(0) space based entirely on the separation properties of the generators of *any* generating set and the corresponding pair-wise commuting products. Moreover the construction of the strict fundamental domain will come from a sequence of a logical moves which at every stage attempts to correct for the reason why a naturally defined subset of X is not a strict fundamental domain. That is we attempt to answer all three questions raised above in the special case that G is a right-angled Coxeter group.

The action of a right-angled Coxeter group on its Davis complex (see Chapter 2 which starts on page 32) has a strict fundamental domain almost by definition. However, if we can look away from the actual definition of the Davis complex and write down some axioms (see 2.2.8) related purely to the nature of the action of the generating set, it is not difficult to isolate exactly why any action that looks like the Davis complex action has a strict fundamental domain. On the short list of these axioms are as follows:

1. The fixed point set of any generator separates the Davis complex into exactly two components.

2. The action of a generator swaps these two components.

3. The action of the link (see page 9) preserves these two components.

4. The fixed point set of every generator in the anti-link of a given generator (see page 10) lies in the same component of the complement of the fixed point set of the given generator.

These four properties guarantee that a given action has a strict fundamental domain. However, they are in some sense artificial. For instance, if one were to take a different generating set for a given right-angled Coxeter system (RACS) none of these properties would hold in general, however the *action is the same*. Thus if we hope to describe a condition which would guarantee that a given action has a strict fundamental domain, the list above is not a good place to start. However, we want to be able to write down an axiom that will eventually allow us to end up with these properties after some manipulation. For instance, in the Davis complex, we know that if we change the generating set to a different generating set, then by results by Radcliffe [Rad01] and Tits [Tit88] that we can get back to the original generating set by applying certain automorphisms (actually in this thesis these results are never actually formerly assumed, see below).

The general plan of attack is as follows. The real work really starts in Chapter 3 where a definition of a right-angled Coxeter pair (RACP) is given. This is a condition which guarantees the existence of a strict-fundamental domain for any geometric action of any right-angled Coxeter group (RACG) on a CAT(0) space.

Then in chapter 4 we proceed to finagle the given generating set into one (by steps that do not change the orbit structure of the original action) that satisfies properties listed just above (actually, the first one listed about having *two* components will never hold, but we will have 2 *kinds* of components that matter, while other kinds of components can be ignored after an inductive procedure, see Chapter 5). The process which allows us to say something along the lines of 3 is called Turning while what allows us to say something along the lines of 4 is known as Folding.

After a technical chapter where we set up the process which allows us to ignore certain kinds of components (which lie a finite Hausdorff distance away from a given fixed point set) we finish the proof in Chapter 6. By this time, we will essentially be able to give our final proofs in a way similar to what we would give in the much simpler situation outlined above in the Davis complex.

Applications for these results can be divided into two separate types:

1. Specifying the space to be the Davis complex $\Sigma(W, S)$ and examining the action of W while focusing on a new generating set S' allows one to get insight into results pertaining to rigidity of right-angled Coxeter groups (as in [Rad01]) as results about the automorphism group of a right-angled Coxeter group (as in [Tit88]). (For a detailed account of

this application see [O'B08])

2. The conditions a weak enough to allow us to gain insight into the kinds of spaces that a particular right-angled Coxeter group can act geometrically. Moreover it also gives us insight into exactly how the action can look on these spaces. For instance, it shows that for each special subgroup that we have a convex subspace on which this special subgroup acts.

The reader should be aware that the author uses the old notation for "subset of" in the sense that $A \subset B$ additionally means that $A \neq B$. The author also refers to what is commonly referred to as a spherical subgroup as simply a *flag* (more on this in Definition 1.1.6). We also often commit the common mathematical notational sins of writing the element in place of a singleton set and identifying a function by its image (especially for a path).

Part I

Review

Chapter 1

The Basics

We first present some basic material that will be considered review. This material will be divided into two parts. The first part is algebraic, and it deals with right-angled Coxeter groups. The second part is geometric, and it deals with group actions on CAT(0) spaces and the Davis complex.

1.1 Right-Angled Coxeter Groups

In this chapter, we attempt to gather all of the basic facts about right-angled Coxeter groups (RACG) that will be used. This section can be skipped by anybody who has had experience with them and can hopefully be used as a reference for those who do not. The typical references for the theory of Coxeter groups are [Hum97] and [Bou02]. There is also a book available from Mike Davis, [Dav08], which presents things from a more geometric point of view. We first start with the definition of a general Coxeter group. In this thesis we will focus primarily on the right-angled versions of these.

Definition 1.1.1. Let I be a well ordered set (usually assumed to be $\{1, 2, \ldots, n\}$ for some integer n). A *Coxeter system* is a pair (W, S) with $S = S(I) := \{ s_i \mid i \in I \}$ and W a group (known as a *Coxeter group*) with a presentation of the form

$$\left\langle S \mid (s_i s_j)^{m(i,j)} = 1 \right\rangle ; \tag{1.1}$$

where m is a function from $I \times I$ to the non-negative integers with $m(i, j) = 1$ if and only if $i = j$. If $m(i, j) = 0$ (resp. $\neq 0$) we say that i and j are unrelated (resp. related). If for all related pairs, i and j, we have that $m(i, j) = 2$, we say that the Coxeter system is *right-angled*. If a Coxeter system is right-angled, we also call the corresponding group right-angled.

In this paragraph (W, S) is a right-angled Coxeter system (RACS) over I. If i and j (in I) are related, we use the notation $i \circ j$. Since all elements of S have order two, $i \circ j$ implies s_i and s_j commute [1]. The converse of this statement is also true and will follow from many of the results quoted below, for instance see Proposition 1.1.4(iv). Associated to (W, S), there is a graph, $\Gamma(W, S)$,

[1]See the index of notation on page 133 for more of this sort of notation

defined with vertex set I and edge set

$$\{\,\{i,j\}\ \mid\ i \circ j\,\}\,. \tag{1.2}$$

On the other hand, given any finite graph Γ, we could define a RACS with I in bijection with the vertex set of Γ with the function m defined to correspond with two times the adjacency matrix plus the identity matrix of the graph Γ plus 1 (i is not adjacent to i).

The following definition gives us a way to characterize to what extent a Coxeter graph (defined only for RACS here) determines its corresponding Coxeter system uniquely.

Definition 1.1.2. Two Coxeter systems are *isomorphic* if the corresponding Coxeter groups are isomorphic. Two isomorphic Coxeter systems are *equivalent* if their Coxeter graphs are also isomorphic (as graphs).

Thus two Coxeter systems are equivalent if there exists an isomorphism of the group which takes one's set of generators to the others. A result of Radcliffe's ([Rad01] or [Rad03]) tells us that if two RACS's are isomorphic, then they are also equivalent, *i.e.*, a RACG is uniquely determined by its graph (up to isomorphism). This result is not true in simple cases for non-right-angled Coxeter groups. (For instance consider the dihedral group of order 16.) Implicit in the results of this thesis is a stronger notion of this rigidity result. Indeed if we replace all occupancies of an abstract space X to the Davis complex we get a geometric proof of rigidity, see [O'B08] for a shorter, more explicit description of this.

Let $\omega = s_1 \cdots s_m$ be a word in (W, S). We define $\ell_S(\omega) = \ell(\omega) = m$. We call this the *length of the word* ω. For $w \in W$, define

$$\ell(w) = \min \{ \ell(\omega) \mid \omega \text{ is a word representing } w \} .$$

We call this the *word length* of w. If a word, ω, represents w and satisfies $\ell(\omega) = \ell(w)$, then we say that the word ω is a *reduced word* or simply *reduced*. Along the same lines, we say that $w_1, \ldots, w_n$ is a *geodesic representative* of w if $w = w_1 \cdots w_n$ and $\ell(w) = \ell(w_1) + \cdots + \ell(w_n)$. Abusing notation, we will call $w_1 \cdots w_n$ a geodesic representation of w whenever $(w_1, \ldots w_n)$ is. Thus a word is a geodesic representation of w if and only if the word is a reduced word representing w.

For $w \in W$,

$$\daleth(w) := \{ i \in I \mid \ell(s_i w) < \ell(w) \} .$$

With the deletion condition below, we can interpret $\daleth(w)$ as corresponding to the set of generators that a word representing w can start. Also, we define

$$\pounds(w) := \{ i \in I \mid s_i \text{ appears in all reduced words which represent } w \} .$$

Below we will see that $\pounds(w)$ could have been defined with the word 'some' replacing the word ‘all’.

The bread and butter of Coxeter groups is the famous Deletion Condition. For a group W generated by involutions S it can be shown to be equivalent to the fact that (W, S) is a Coxeter system, *i.e.*, that it has a presentation of the form 1.1.

Lemma 1.1.3 (The Deletion Condition [2] [Hum97], [Dav08], [Bou02])**.** *Suppose that $s_1 \cdots s_n$ is a word in S with $\ell(s_1 \cdots s_n) < n$. Then there exists $1 \leq i < j \leq n$ so that*

$$s_1 \cdots s_i \cdots s_j \cdots s_n = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_n.$$

(where the hat denotes omission). □

A special subgroup of (W, S) is a group generated by $T \subseteq S$. We denote it by W_T or $W_{I(T)}$ where $I(T) \subseteq I$ is the corresponding index set.Conjugates of special subgroups are called *parabolic subgroups*. The same references apply for this proposition. They are all basic consequences of the exchange condition.

Proposition 1.1.4 (Special Subgroups)**.** *For $T, T_1, T_2 \subseteq S$ the following hold:*

(i) *For any $w \in W_T$, $\pounds(w) \subseteq I(T)$.*

(ii) *(W_T, T) is a* RACS.

(iii) $W_{T_1 \cap T_2} = W_{T_1} \cap W_{T_2}$.

[2] or Exchange Condition

(*iv*) *The word length in* (W_T, T) *and the word length as defined in the* (W, S) *restricted to* W_T *coincide.*

(*v*) *If* W_T *is finite, then there exists a unique element (denoted* w_T*) of* W_T *of maximal word length.* $\square$

With the exception of (*ii*) (which by definition only holds for RACS), these hold for all Coxeter groups. While we are at it we state some more basic facts about RACG's.

Proposition 1.1.5. *For any* RACS (W, S)

(*i*) *For any* $s \in S$*, the centralizer of* s *is a special subgroup of* W*.*

(*ii*) *The abelianization of* W *is* $\mathbb{Z}_2^{|S|}$ *and the abelianization map is the obvious one sending* s_i *to* e_i*, the* i^{th} *standard basis vector of* $\mathbb{Z}_2^{|S|}$*.*

(*iii*) *The set of elements of* S *that appear in any reduced word representing a single element of a* RACG *is the same.* $\square$

The special subgroup in question in part (*i*) is generated by the set of elements of S whose corresponding vertices are adjacent to s as well as s itself in the defining graph of (W, S). It is possible to argue that this part of the lemma is the biggest reason why this thesis deals nearly exclusively with RACG's.

Definitions 1.1.6. A *flag* [3]is a set $\mathfrak{f} \subseteq I$ with $W_{\mathfrak{f}}$ finite. We say that i is flag to w if $i \circ \pounds(w)$. We refer to elements of $W_{\mathfrak{f}}$ as *flag elements* (we may abuse

[3]In the literature flags are typically called spherical subsets.

terminology and call the corresponding element of a group a flag) and words representing flag elements as *flag words*. We let $w(\mathfrak{f})$ or $w_{\mathfrak{f}}$ denote the unique element of $W_{\mathfrak{f}}$ of longest length. Lastly, we call $w(\mathfrak{f})$ an *i-flag* if $i \in \mathfrak{f}$. We denote the set of all flags of (W, S) as $\mathcal{S}(W, S)$ or $\mathcal{S}$.

Notice that $\mathcal{S}$ forms a simplicial complex with vertex set I and $J \subseteq I$ spanning a face if and only if J is a flag. This simplicial complex plays a large role in much of the literature on Coxeter groups and it is called the *nerve* of a Coxeter system. For instance, it is possible to read of the compactly supported cohomology of (W, S) simply by the topology of its nerve. For us the most important part of this simplicial complex structure is the associated poset, $(W, <)$, which is induced by inclusion. Even this will be used mostly for notational convenience. For a given flag, $\mathfrak{f}$, we denote the set $\mathcal{S}_{\star\mathfrak{f}}$ as the set of flags that are $\star\,\mathfrak{f}$ ($\star \in \{<, \leq, >, \geq\}$). In the simplicial complex, $\mathcal{S}_{<\mathfrak{f}}$, $\mathcal{S}_{\leq\mathfrak{f}}$, $\mathcal{S}_{>\mathfrak{f}}$, and $\mathcal{S}_{\geq\mathfrak{f}}$ correspond respectively to the set of proper faces, the set of faces, the link, and the star of the face $\mathfrak{f}$. Also notice that by definition, $\mathcal{S}_{>i}$ is the set of i-flags.

Definition 1.1.7. We say that a RACS is f-irreducible whenever the compliment of the defining graph has no isolated point. Otherwise, we say that the RACS is f-reducible.

Playing with definitions we see that a RACS, (W, S) is f-reducible if and only if the group W can be written as $W = W' \times W_{\{i\}}$ where $s_i \in \mathfrak{Z}(W)$. Also, we can see that (W, S) is f-irreducible if and only the center of W is trivial.

1.1.1 More basic facts

Here are some more basic facts that are related to right-angled Coxeter groups and their relation with other Coxeter groups.

(a) If $i \bullet j$, then $\langle s_i s_j \rangle$ is infinite. This is a restatement of the following two facts: the restricted length function to a special subgroup coincides with the original, and the group $\left\langle s_i, s_j \mid s_i^2, s_j^2 \right\rangle$ is the *infinite* dihedral group.

(b) A right-angled Coxeter group is abelian if and only if it is finite.

(c) A Coxeter group is right-angled if and only if the commutator is abelian.

(d) If w has finite order, then $w^2 = 1$ (see Proposition 2.2.6 and its corollary).

(e) RACG's are distinguished from other Coxeter groups by (d). That is if every finite order element of a Coxeter group is two, then the Coxeter group is right-angled.

(f) A right-angled Coxeter group has a non-trivial center if and only if it can be written as $W' \times \mathbb{Z}_2$ (see comments below the definition of f-irreducible).

(g) $S(\mathsf{lk}(i) \cup \{i\})$ [4] generates the centralizer of s_i (this is just a restatement of part of Lemma 1.1.5).

[4] $\mathsf{lk}(i)$ is the graph theoretical *link* in the Coxeter diagram, i.e. the set of vertices of distance one away from i in $\Gamma(W, S)$

The group $W_{\mathsf{lk}(i)}$ will play an important part in our study of right-angled Coxeter pairings. We call this group *the link group of* s_i. We also use the notation $\mathsf{c}(i) := \mathsf{lk}(i) \cup \{i\}$ (this is known as the graph theoretical *star* of a vertex) and $\mathsf{kl}(i)$ for the set of generators $I \setminus \mathsf{c}(i)$. The group that $\mathsf{c}(i)$ generates has already been pointed out to be the centralizer of s_i and the group that $\mathsf{kl}(i)$ generates will be called the *anti-link of i*. It will also play an important (but dual) role.

Notice that several things we have said so far amount to

$$\mathrm{C}_W(s_i) = W_{\mathsf{c}(i)} = W_{\mathsf{lk}(i)} \times W_i = W_{\mathsf{lk}(i)} \times \langle s_i \rangle$$

where $\mathrm{C}_W(s_i)$ is the centralizer of s_i in the group W.

1.1.2 Automorphisms of Right-Angled Coxeter groups

It follows from general results proven in [FR40] that there are exactly four types of outer automorphisms of a right-angled Coxeter group. The table below defines these. (We will refer to lifts of these automorphisms to the full automorphism group by the same names.)

Elements of the form φ_σ are known as *graph symmetries*, of the form $\mathrm{Tv}_{(i,j)}$ are known as *basic transvections*, and of the form $\mathrm{Pc}_{(s,T)}$ are known as *basic partial conjugations*. Products of basic transvections (resp. partial conjugations) are

Name	**Definition**	**Condition for Existence**
φ_σ	$s_i \mapsto s_{\sigma(i)}$	$\sigma \in \mathrm{Aut}(\Gamma(W,S))$.
$\mathrm{Tv}_{(i,j)}$	$s_i \mapsto s_i s_j, s_k \mapsto s_k$ for $k \neq i$	$\mathsf{c}(i) \subseteq \mathsf{c}(j)$
$\mathrm{Pc}_{(s,T)}$	$t \mapsto sts$ for $t \in T$	$T \subset \pi_0(\Gamma(W,S) \setminus \mathrm{St}(s))$

Table 1.1: In the table $T \subset S$, $W_{\mathsf{c}(*)}$ is the centralizer of s in W, and $\mathrm{St}(s)$ is the star of s in $\Gamma(W,S)$. Also, maps on generators are understood to induce the obvious map on W.

known as *transvections* (resp. *partial conjugations*). We will refer to lifts of these various outer automorphisms to the full automorphism group by the same names. The set of partial conjugations forms a subgroup of $\mathrm{Out}(W)$ referred to as $\mathrm{Out}^\circ(W)$. At the same time the set of graph symmetries and transvections also forms a subgroup, and this group is known as $\mathrm{Out}^1(W)$.

In [Tit88], it is shown that $\mathrm{Out}(W)$ splits as

$$\mathrm{Out}(W) = \mathrm{Out}^\circ(W) \rtimes \mathrm{Out}^1(W)$$

(see [O'B08] for a geometric proof of this). The group $\mathrm{Out}^1(W)$ can be realized as the automorphism group of $\mathcal{S}(W,S)$ and as such it is finite. For a presentation of $\mathrm{Aut}^\circ(W)$ (the inverse image of $\mathrm{Out}^\circ(W)$ in $\mathrm{Aut}(W)$) see [M̈98].

1.2 (Double) Coset Representatives

In this section we discuss minimal double coset representatives and two ways in which to use these representatives to get an elementary normal form for $w \in W$ known as a *flagged centralizer decomposition*. As already pointed out, $\daleth(w)$ has a corresponding set of generators that indicate which letters a word representing w can start with. That is $i \in \daleth(w)$ if and only if there exists a $w_i \in W$ with $s_i w_i$ a geodesic representation of w. Also, by definition we have that

$$\daleth(w^{-1}) = \{\, i \in I \mid \ell(ws_i) = \ell(w) - 1 \,\}\,.$$

And we point out that if $1 \neq w$, then $\daleth(w) \neq \emptyset$. Moreover $\daleth(w)$ is a flag for all w; for i, $j \in \daleth(w)$ implies that $i \circ j$. ($\daleth$ can obviously be defined for any Coxeter group, and the fact that it corresponds to a finite groups is always true. It, however, takes a little bit more work when the group is not right-angled, see [Dav08].) We record what we just showed for future reference.

Lemma 1.2.1. *For every $w \in W$, $\daleth(w)$ is a flag.* □

1.2.1 Double Cosets

With these definitions and lemmas out of the way, we begin to discuss double cosets. We give the relevant definitions for those who are not familiar with them. Given a group G and subgroup H_1 everybody in their first algebra learns how to form the set of left cosets G/H_1 (many books do not use this notation unless the group H_1 is normal). Later, we learned a fancy way of stating the definition of this; it is simply the set of orbits under the obvious left action of H_1 on the group G (we give the relevant definitions pertaining to group actions in the next section). Similarly, one can define the set of right cosets as the set of orbits under the obvious right action of H_1 on G, and when we do not care when we are talking about either left or right cosets, we call them cosets. Double cosets are simply the next step. If we have two subgroups, H_1 and H_2 of G (you knew there was a reason that the subscript "1" was used), we can define an action of $H_1 \times H_2$ on G by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$ where the multiplication here takes place in G. A double coset (orbit) is written as $H_1 g H_2$. We denote the set of orbits of this action as $H_1 \backslash G / H_2$. We can think of the set of left (resp. right) cosets of H_1 in G as the set $H_1 \backslash G / \{1_G\}$ (resp. $\{1_G\} \backslash G / H_1$). As with the set of cosets, we can define a coset representative of $H_1 g H_2$ as any element of it. Hence given a particular set of double coset representatives, we can right any element $w \in G$ as $w = h_1 g h_2$ where $H_1 g H_2 = H_1 w H_2$ and $h_1 \in H_1$ and $h_2 \in H_2$ (unlike usual coset representatives, this decomposition is not unique for a fixed g).

We now start to put together the first part of this section, on special subgroups, with these double cosets.

Proposition 1.2.2. *Any double coset has a unique representative of minimal word length.*

Proof. While this is not difficult to prove in general (see [Dav08]), it is even easier in the right-angled case. We give this proof only to help the reader familiarize themselves with the idea of double cosets and right-angled Coxeter groups. Let w_1, w_2 be representatives of the double coset in question with smallest length. First, we notice that for any $s \in W_T$ and $s' \in W_T$, we must have that $s \notin S(\daleth(w_i))$ and $s' \notin S(\daleth(w_i^{-1}))$ $(i = 1, 2)$. This is because if either were not the case, we would have a shorter double coset representative. Now, we must have $(h_1, h_2) \in H_1 \times H_2$ with $h_1 w_1 h_2^{-1} = w_2$. By what has just been said, h_1 must be flag to w_1 and $\ell(h_1 h_2^{-1}) = 0$. Hence $w_1 = w_2$. $\square$

Given T_1, $T_2 \subseteq I$, we denote the set of minimal double coset representatives as ${}^{T_1}W^{T_2}$. If $w \in {}^{T_1}W^{T_2}$, we say that w is (T_1, T_2) reduced. If T_1 (resp. T_2) is empty, we denote ${}^{T_1}W^{T_2}$ as ${}^{T_1}W$ (resp. W^{T_2}) and call its elements [left] T_2 (resp. [right] T_1) reduced. Notice this notation makes sense since ${}^{\emptyset}W^{\emptyset} = W$ (the original RACG).

This is now obvious:

Corollary 1.2.3. *We use the same notation as above.*

(i) *Every $w \in W$ has a geodesic representation $w = w_1 w' w_2$ where $w_1 \in W_{T_1}$, $w_2 \in W_{T_2}$, and $w' \in {}^{T_1}W^{T_2}$.*

(ii) *The decomposition in (i) is unique whenever $T_1 \cap T_2 = \emptyset$.*

(iii) *For any $w' \in W$, $w' \in {}^{T_1}W^{T_2}$ if and only if*

$$\urcorner(w) \bigcap T_1 = \emptyset \quad and \quad \urcorner(w^{-1}) \bigcap T_2 = \emptyset.$$

□

The application below illustrates a completely typical application of double cosets.

Corollary 1.2.4. *If ws_jw^{-1} commutes with s_i, then w is in the $W_{\mathsf{c}(i)}1_W W_{\mathsf{c}(j)}$, the double coset of the identity.*

Proof. Write $w = w_1w_2w_3$ where $w_1 \in W_{\mathsf{c}(j)}$, $w_2 \in {}^{\mathsf{c}(j)}W^{\mathsf{c}(i)}$ and $w_3 \in W_{\mathsf{c}(i)}$. We must show that $w_2 = 1$. Thus we must have that $w_2s_2w_2^{-1}$ commutes with s_1. By properties of w_2 we must have that $\pounds(w_2s_jw_2^{-1}) = \pounds(w_2) \cup \{s_j\}$ and by Lemma 1.1.5 we know that s_i commutes with every letter in $\pounds(w_2)$. Now the above lemma says that w_2 cannot start with any letter which commutes with s_1 and thus $w_2 = 1_W$ as desired. □

Of particular use is the case that $i = j$ which guarantees that the only element in the conjugacy class of a given generator which commutes with that generator is the generator itself.

Often, we will use these results in the context of the centralizer of a generator or a spherical subgroup. In this vein, we get two important decompositions that we will apply to elements of W.

1.2.2 Decompositions

Flagged Decomposition Let $w \in W$. We will first describe the *flagged decomposition of* w. We inductively define a set $\{ w_1, \ldots, w_m \}$. First, define $w_1 := w$ and then inductively define $w_{i+1} := w(\mathfrak{f}_i)w_i$ where $\mathfrak{f}_i := \urcorner(w_i)$. (Note that $\mathfrak{f}_i$ are indeed flags by comments made after first defining $\urcorner$.) Define $\ell_{\mathfrak{f}}(w)$ as the first value, m, with $w_m = 1$. For the most part, we are mostly interested in the terms $w(\mathfrak{f}_i)$ $(0 \leq i \leq \ell_{\mathfrak{f}}(w))$ and their corresponding flags. These give a factorization of w as $w = w(\mathfrak{f}_1) \cdots w(\mathfrak{f}_m)$. This factorization is determined uniquely by w and, in turn, the sequence of flags, $(\mathfrak{f}_1, \ldots, \mathfrak{f}_m)$, uniquely determine w. Collectively, this sequence of flags will be called the *flag decomposition* of w.

Centralizer Decomposition We define the centralizer decomposition of $\omega = s_1 \cdots s_m$ where ω is a reduced[5] word representing an element of w. Define $a_o = 1$ and inductively define a_i as the first letter (reading left to right) with $a_i \bullet a_{i+1}$. We call $s_{a_i} \cdots s_{a_{i+1}}$ a (centralizer decomposition) *syllable* of the decomposition. We define $\ell_C(\omega)$ as the number of syllables of the decomposition. For $w \in W$, we say that a sequence of reduced words *is a centralizer decomposition of* w if there is some reduced word representing w with the sequence of words a centralizer decomposition of w.

[5]Notice that, strictly speaking, we could define the centralizer decomposition of any word, reduced or not. The resulting centralizer decompositions of various sub-words can then used to determine if the original word was reduced or not

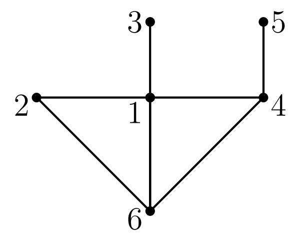
Remark 1.2.5. Notice that a flag decomposition is a sequence of flags while, in contrast, a centralizer decomposition is a sequence of words.

Remark 1.2.6. From now on, we will think of the a_i only when discussing centralizer decompositions. Thus, if $s_1 \cdots s_{a_2-1} s_{a_2} \cdots \cdots s_{a_r} \cdots s_m$ is a centralizer decomposition of $s_1 \cdots s_m$, then the a_i are understood to have the meaning above.

Flagged Centralizer Decomposition A *flagged centralizer decomposition* of $w \in W$ is a centralizer decomposition of w where each syllable is a flag word. We now show that such a decomposition exists for every $w \in W$. Roughly, the construction comes from a flag decomposition of w, and then by flipping around some of the letters in each flag word, we also create a centralizer decomposition. We now give the construction. Suppose that $\ell_{\mathfrak{f}}(w) = m$ and let $(\mathfrak{f}_1, \ldots, \mathfrak{f}_m)$ be the flag decomposition of w. Pick any element of $w(\mathfrak{f}_m)$ and set it equal to a_m. Now, assuming that $a_i \in \mathfrak{f}_i$ has been defined, we describe how to define a_{i-1}. We simply let a_{i-1} be an element of $\mathfrak{f}_{i-1}$ with $a_{i-1} \bullet a_i$. Such a letter exists by definition of the flag decomposition. We now define a word, ω, as the product $\overline{w(\mathfrak{f}_1)} \ldots \overline{w(\mathfrak{f}_m)}$ where $\overline{w(\mathfrak{f}_i)}$ is any word representing $w(\mathfrak{f}_i)$ with the restriction being that the word starts with s_{a_i}. The centralizer decomposition of ω is clearly a flagged centralizer decomposition of w.

Remark 1.2.7. When we do not want something as specific as a flagged centralizer decomposition, we will have occasion to use the notion of a (general) centralizer decomposition. However, the primary purpose for the notion of a flag decomposition was to give a clear account of the existence of a flagged centralizer decomposition.

Example 1.2.8. Consider the RACG determined from the following graph.



Suppose an element $w \in W$ is represented by the word

$$s_1 s_2 s_4 s_5 s_6 s_3 s_2 s_6 s_5 s_4.$$

The flag decomposition of w is

$$(\{1,2\}, \{4,5\}, \{6\}, \{3\}, \{2,6\}, \{5,4\}).$$

Thus, $\ell_{\mathfrak{f}}(w) = 6$. Following the procedure above, we get the following flagged centralizer decomposition of w:

$$\underline{s_2 s_1}\ \underline{s_5 s_4}\ \underline{s_6}\ \underline{s_3}\ \underline{s_2 s_6}\ \underline{s_5 s_4}$$

1.3 CAT(0) Spaces and Geometric Actions

What manifolds do for calculus CAT(0) spaces do for geometry. CAT(0) spaces are the spaces with the least amount of structure in which one can use the techniques that resemble those used in classical geometry. All of the information here is presented in the book [BH99]. We do not actually give the definition of a CAT(0), but we only state the properties that will be needed. We will see throughout most of this section that a majority of what is proven in this document could be proven for uniquely geodesic spaces. This is because order two elements do not care whether they act on a CAT(0) space or a unique geodesic space.

The first two sections of this chapter are extremely basic. They talk about metric spaces and group actions in general. We only do this to point out some unusual terminologies. (Not unusual for geometric group theorists however!)

1.3.1 Metric Spaces

An isometry is a surjective map of metric spaces which preserves the distance function. The set of isometries from a metric space X to itself forms a group. This group is the *isometry group* of X and it is denoted $\mathrm{Isom}(X)$.

Definition 1.3.1. For a metric spaces (X, d_X) and (Y, d_Y):

- A subset of X is *bounded* if it is contained in a ball $B_r(x)$ ($r < \infty$).
- The *product metric* on $X \times Y$ is the function $d_{X\times Y}$ defined by (for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$)

$$d_{X\times Y}((x_1, y_1), (x_2, y_2))^2 := d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2 .$$

  It is elementary to see that the product metric is indeed a metric on $X \times Y$.
- A function is *proper* if the inverse image of every compact set is compact. A metric space is *proper* if the corresponding distance function is proper.

1.3.2 Group Actions

To establish notation, we review group actions. The main theme of geometric group theory is the study of the structure of a group from the geometry of a space in which it acts. A *(left) group action* of a group G on a set X will always be thought of as homomorphism

$$\rho\colon G \to \mathrm{Bij}(X)$$

where $\mathrm{Bij}(X)$ is the group of bijections of X. (Occasionally, for notational convenience, we will also deal with right actions, these are anti-homomorphisms into $\mathrm{Bij}(X)$). We denote this by $G \overset{\rho}{\curvearrowright} X$ (or $G \curvearrowright X$). This is to be read as G acts on X via ρ (or G acts on X). For instance, the first two examples one usually sees of finite non-abelian groups, the dihedral groups and the symmetric groups, are defined in terms of group actions. Often, one insists that X has further structure and the image of G in $\mathrm{Bij}(X)$ is compatible with this structure. For example, if X is a metric space, we will often require $\rho(G) \subseteq \mathrm{Isom}(X)$, and we say that G acts X by isometries. Most of the time, we write $\rho(g)(x)$ as $g \cdot x$. The one exception occurs in Section 4.2. This is because in that section we will have two relevant group actions. The set of spaces on which a group G acts forms a category with objects the spaces and morphisms

$$\hom(X_1, X_2) := \{\, \varphi\colon X_1 \to X_2 \mid \varphi(g\cdot x) = g\cdot\varphi(x) \,\}.$$

Objects are called *G-spaces*, and maps are called *G-equivariant*.

Example 1.3.2. For any subgroup $H \subseteq G$, the set of cosets G/H is a G-space with $g\cdot(g_2H) = (gg_2)H$.

Definitions 1.3.3. We give our notations for the three main definitions that are related to a group action.

- The *fixed point set* of $S \subseteq G$ is the set

$$\{\, x \in X \mid \rho(g)(x) = x \text{ for all } g \in S \,\}.$$

It is a convex subset of X whenever X is a UGM space. It is denoted X^S_ρ. If

ρ is understood from context, we write this as X^S. If S consists of a single point s, then we denote $X^{\{s\}}$ as X^s.

- The dual notion of the fix point set of a subgroup is that of the stabilizer of a subset A or X. It is the set

$$\{\, g \in G \mid \rho(g)(A) = A \,\}.$$

[6] It is easy to see that it is a subgroup of G, and it is denoted $\mathrm{St}_{\rho(G)}(A)$. If G is understood from context, then we write $\mathrm{St}_{\rho(G)}(A) = \mathrm{St}_{\rho}(A)$. If ρ is understood from context, we write $\mathrm{St}_{\rho(G)}(A) = \mathrm{St}_G(A)$. However, most of the time both are understood from context, and we write $\mathrm{St}_{\rho(G)}(A) = \mathrm{St}\, A$ or $\mathrm{St}(A)$.

- The *orbit* of $A \subseteq X$ is the set

$$\{\, y \in X \mid y = g\cdot a \text{ for some } g \in G \text{ and } a \in A \,\}.$$

It is denoted $G^\rho \cdot A$. We omit ρ if it is understood (there will be several key times where it will not be understood) and denote $G^\rho \cdot \{x\}$ by $G^\rho \cdot x$. The relation $x \sim y$ if and only $y \in G\cdot x$ is an equivalence relation, and $G\cdot x$ is a G-space. (It is interesting to note that the proof of the first statement uses every

[6]This is not to be confused with the *point-wise stabilizer* of a set which is given by

$$\{\, g \in G \mid \rho(g)(a) = a \text{ for all } a \in A \,\}$$

axiom of a group in exactly one non-trivial way, e.g, transitivity corresponds to associativity, reflexivity corresponds to the existence of an identity, and symmetry corresponds to the existence of inverses.) We denote the set of equivalence classes of this equivalence relation (that is the set of orbits) by X/G.

Definitions 1.3.4. An action is *transitive*if for some $x \in X$, $G \cdot x = X$. An action if *faithful*if $\ker(\rho) = 1_G$. An action is *free*if for every $x \in X$, $\mathrm{St}(x) = 1_G$. Of course, a free action is faithful (in a very strong sense).

We record the three most used facts pertaining to group actions. Many seemingly non-trivial statements in many areas of mathematics are simply restatements of them.

Proposition 1.3.5. *The following hold for $G \curvearrowright X$.*

(*i*) $g \cdot X^S = X^{gSg^{-1}}$

(*ii*) $g\,\mathrm{St}(A)g^{-1} = \mathrm{St}(g \cdot A)$

(*iii*) *For any $x \in X$ there is a G-equivariant equivalence*

$$G/\,\mathrm{St}(x) \approx G \cdot x.$$

Proof. Simply rewriting both sides of (*i*) and (*ii*) gives you them. The map inducing the equivalence in (*iii*) is given by

$$g\,\mathrm{St}(x) \mapsto g \cdot x. \qquad \square$$

For the most part, geometric group theorists are concerned with geometric actions. For an action of G on a metric space X, we define a topology G/X induced by the pseudo-metric (it is a metric if the action is proper as in 1.3.6 below):

$$d_{G/X}(G\cdot x, G\cdot y) := d_X(G\cdot x, G\cdot y)\,. \tag{1.3}$$

Definitions 1.3.6. An action of G on X is *cocompact* if X/G is compact in the topology induced by (1.3). A group action is *proper* if for every bounded open set B, the set

$$\{\, g \in G \mid g\cdot B \cap B \neq \emptyset \,\}$$

is finite. An action by isometries is *geometric* if it is both proper and cocompact.

Notice that properness requires that if the order of $g \in G$ is infinite, then $X^g = \emptyset$. Thus, any proper action of a torsion free group is free (and a covering space action).

Remark 1.3.7. For X proper, the following are some equivalent notions of

1. cocompactness:

   (a) There is a positive number K so that given any x, $y \in X$, there exists a $g \in G$ with $d(x, g{\cdot}y) \leq K$.

(b) There exists a compact set K with $G{\cdot}K = X$.

2. properness:

(a) For any compact set K

$$\{\, g \in G \mid g{\cdot}K \cap K \neq \emptyset \,\}$$

is finite.

(b) Counting repetitions, the orbit of any point is locally finite (see Definition 1.3.17 below)

(c) Not counting repetitions, the orbit of any point is discrete and the stabilizer of every point is finite.

(d) The map $G \times X \to X \times X$ defined by $(g, x) \mapsto (g{\cdot}x, x)$ is a proper map.

These are alternate definitions that one may find in the literature. The proof of most of these are trivial. The alternate cocompactness definitions essentially differ from the original only in wording.

1.3.3 Geodesic Metric Spaces

A geodesic metric space is a metric space where one can measure distance by the length of paths.

Definition 1.3.8. A *path* in a topological space X is a continuous function from an interval in $\mathbb{R}$ to X. Occasionally, we will refer to a path by its image. The most important kind of path in this document is a geodesic. A *geodesic* between points x and y is a path which is also an isometry from the interval $[\,0, d(x, y)\,]$ in $\mathbb{R}$ onto its image. A metric space X is a *(uniquely) geodesic metric space* if any two points in X can be connected by a (unique) geodesic. (Uniquely geodesic metric spaces will be called UGM spaces.) A *geodesic ray* is an isometry from a ray in $\mathbb{R}$ to X. A set $C \subseteq X$ of an UGM space, X, is *convex* if for every x and y in C, the geodesic between x and y (denoted $[\,x, y\,]$) is contained in C.

We will not give a precise definition of a CAT(0) space. This is because we will not make use of the actual definition but only of certain properties (listed below). It is not that the definition of a CAT(0) space is difficult — see just about any other article dealing with CAT(0) geometry — but it will simply serve as a distraction from the key properties that will be used in the proof. Some of these are:

Some facts about CAT(0) **spaces**

- CAT(0) spaces are UGM spaces.
- CAT(0) spaces are locally path connected.
- If a geodesic ray starts in a convex set A and it eventually exits A, then it ends up an infinite distance from A. We will refer to this as *weak convexity*.
- The following will allow us to make inductive-like procedures:

  Proposition 1.3.9 ([Rua01]). *Suppose a group G acts geometrically on a* CAT(0) *space. Then, the centralizer of g acts geometrically on* $\mathrm{Min}(g)$ *(see below for definition of* $\mathrm{Min}(g)$*) for all $g \in G$.* $\square$

1.3.4 Actions on CAT(0) and UGM Spaces

Notation 1.3.10. Throughout,

$$|g| := \min_{x \in X} d(x, g \cdot x) \quad \text{and} \quad \mathrm{Min}(g) := \{\, x \in X \mid d(g \cdot x, x) = |g| \,\}.$$

The next few definitions are only relevant for a RACS, (W, S).

Definitions 1.3.11. For a flag $\mathfrak{f}$,

$$X^{\mathfrak{f}} := \bigcap_{i \in \mathfrak{f}} X^{s_i}. \tag{1.4}$$

We will refer to a set as above as a *fixed flag*. We say that an action, $\rho\colon W \to \mathrm{Isom}(X)$, is *properly nested* if for every flag, $\mathfrak{f}$, $X^{\mathfrak{f}} = X^{w(\mathfrak{f})}$.

Written out, the proper nesting requirement reads

$$X^{s_1} \cap \cdots \cap X^{s_m} = X^{(s_1 \cdots s_m)}$$

whenever $\{1, \ldots, m\}$ is a flag.

Definition 1.3.12. For $w \in W$, define

$$\mathcal{T}_w := \pi_0(X \setminus X^w).$$

For a generator, $s_i \in S$, we denote $\mathcal{T}_{s_i}$ as $\mathcal{T}_i$.

Notation 1.3.13. Given $W \curvearrowright X$ and $A \subseteq X$, we use the following notations:

- $\mathsf{in}(A) := \{ i \in I \mid X^{s_i} \subseteq A \}$
- $\mathsf{mt}(A) := \{ i \in I \mid X^{s_i} \cap A \neq \emptyset \}$.

Note that if $W \curvearrowright X$ is proper, then $\mathsf{mt}(x)$ is a flag.

The following proposition is true in general only for CAT(0) spaces (and not for UGM spaces in general).

Proposition 1.3.14 (Elliptic Elements, CAT(0) space)**.** *Every action of a finite group on a* CAT(0) *space fixes a point.* $\square$

Thus for elliptic elements g, $\mathrm{Min}(g) = X^g$. The idea of the proof is essentially to take the convex hull of the (finite) orbit of H and take its barycenter. This barycenter turns out to be the fixed point of the action of the finite group. While this statement needs the CAT(0) geometry, the following holds for any uniquely geodesic metric space. The key difference is that in a uniquely geodesic metric space, the barycenter of a geodesic makes sense, but the barycenter of a larger region does not. We give a proof only to stick with our theme of using as little CAT(0) geometry as possible.

Proposition 1.3.15 (Elliptic Elements, UGM space)**.** *Suppose that X is a* UGM *space. If (W, S) is a finite* RACS *(i.e.a direct product of cyclic groups of order 2), then there is a point $x \in X$ fixed by W, i.e., $X^W \neq \emptyset$.*

Proof. We give the proof only because it is too easy to skip. Since the fixed point set of an $s \in S$ is convex and every other $s' \in S \neq s$ leaves X^s invariant, we only need to verify the result for $|S| = 1$. The action of s exchanges the endpoints of the geodesic $[\, x, s{\cdot}x\,]$. Thus, by uniqueness of geodesics, s fixes the entire geodesic, and the midpoint of this geodesic is fixed. ☐

Proposition 1.3.16. *Suppose that C is a closed, convex, and g-invariant subset of X. Then $\mathrm{Min}(g) \cap C \neq \emptyset$.* ☐

Definition 1.3.17. For a topological space X, a collection $\mathcal{C}$ of subsets of X is *locally finite* if every compact subset of X meets only a finite number of sets of $\mathcal{C}$.

Here are some trivial examples to get a loose understanding of this.

Example 1.3.18. The set of points of a discrete space is locally finite. The set of points in a path connected space consisting of more than two points is not locally finite. The set of simplicies in a simplicial complex is locally finite if and only if the simplicial complex is locally compact. Any finite collection of sets is locally finite.

Lemma 1.3.19. *Suppose $G \curvearrowright X$ properly. Then, if G is generated by the finite set S, then the collection*

$$\{\, g \cdot X^s \mid g \in G \text{ and } s \in S \,\}$$

is locally finite.

Proof. Direct consequence of the definition (actually the definition about compact sets which is given after the definition) of a proper action and the fact that $w \cdot X^s = X^{wsw^{-1}}$. $\square$

The following easy lemma and its corollary will be used several times.

Lemma 1.3.20. *Let X be a geodesic metric space. Suppose that $\mathcal{C}$ is a locally finite collection of closed convex separating subsets of X. Suppose a map $\varphi\colon X \to X$ has the following properties:*

- *φ preserves $\mathcal{C}$ and is an isometry there.*

- *If A_1 and A_2 are components of*

$$X \setminus \bigcup_{C \in \mathcal{C}} C$$

and $\varphi(A_1) = A_2$, then $\varphi|_{A_1}$ is an isometry.

Then φ is an isometry of X.

Proof. For $x, y \in X$, break up the geodesic $[\,x, y\,]$ as it goes through sets in $\mathcal{C}$ to get the result. □

The following Corollary will be used for instance in the proof the main theorem.

Corollary 1.3.21. *In the situation above, let*

$$C := X \setminus \left(A \coprod B\right).$$

Suppose that an isometry ρ preserves A and B and fixes C point-wise, then for any $a \in A$ and $b \in B$, $d(a, b) = d(a, \rho(b))$.

Proof. Define the map φ as ρ on $A \cup C$ and as the identity on $B \cup C$. By the lemma, this gives an isometry of X which takes the geodesic $[\,a, b\,]$ to the geodesic $[\,a, \rho(b)\,]$. □

Chapter 2

The Davis Complex

This is the final chapter of introductory material before getting started with the main part of this document. It defines the last main object of study of Part I. Also, in some sense, the nicest CAT(0) space which a Coxeter group can act geometrically.

2.1 Strict Fundamental Domains

Tessellations are one of the easiest ways to motivate the subject of group theory as a whole. These beautiful objects are familiar to everybody in the forms of wallpaper patterns and Eschar prints (see Figure 2.1). Even students at elementary points of their math education study wallpaper groups. Strict fundamental domains are intimately related with the concept of a tessellation.

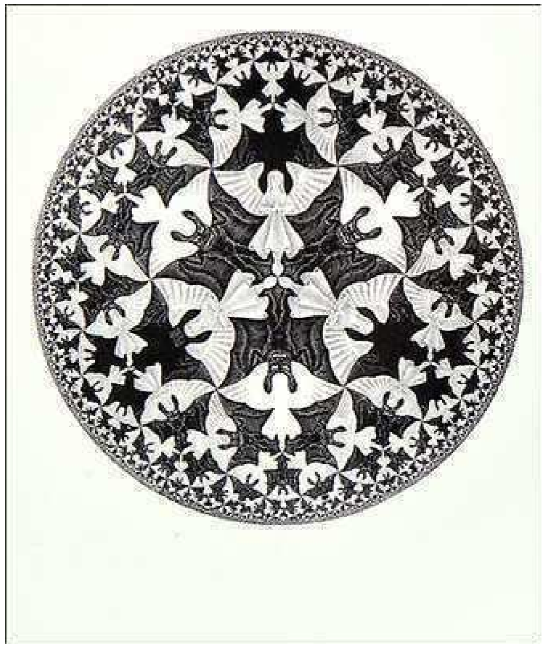


Figure 2.1: Circle Limit IV by M.C. Eschar

Throughout this section, we suppose that G is a group acting on a metric space X by isometries. K will always denote a closed subset of X.

Definitions 2.1.1 (Strict Fundamental Domain)**.** A closed set $K \subseteq X$ is a *strict set* for an action, $G \curvearrowright X$, if given any $x \in K$ and $g \in G$, if $g \cdot x$ is also in K, then $g \cdot x = x$. That is to say every orbit meets K at most once.

K is a *fundamental domain*for the action, if given any $x \in X$ there is a $g \in G$ with $g \cdot x \in K$. That is to say every orbit meets K at least once. A closed set K is a *strict fundamental domain* if K is both a strict set and a fundamental domain.

Notice that every group action has a strict set, namely a single point and a fundamental domain, namely the space itself. Moreover a Zorn's lemma argument shows that there is a maximal strict set and a minimal strict set. However, most actions do not have a strict fundamental domain. For instance, the rotation of a square by 180 degrees does not have a strict fundamental domain.

Example 2.1.2. According to Bass-Serre theory, see [Ser89], any action of a group generated by elements of finite order on a tree has a strict fundamental domain. In this case, the strict fundamental domain determines an amalgamated product decomposition of the group with the amalgamating groups are finite. Conversely, if the original group has this decomposition, then there is a tree in which the group acts geometrically.

Proposition 2.1.3. *Suppose G acts geometrically on a* UGM *space X with a strict fundamental domain K whose translates are locally finite. Then*

(*i*) *K is convex, and*

(*ii*) *the the quotient X/G is isometric to K.*

Proof. We prove (*ii*) first. We define a map r by sending x to the unique point of $W \cdot x$ which meets K.

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "r"] & \\ X/G \arrow[r, "\hat{r}"] & K \end{tikzcd}$$

The map r induces the bijection $\hat{r}$ as shown in the diagram. Since X/G is compact and, as a subspace of X, K is Hausdorff, we need only show that $\hat{r}$ is continuous. This is equivalent to showing that r is continuous (since X/G has the quotient topology). Since the collection $\{w \cdot K\}_{w \in W}$ is locally finite by assumption, we can check continuity of r by checking the continuity of r when restricted to any translate $w \cdot K$. However on this restriction r is simply translation by w, which is an isometry by assumption and hence is continuous.

For part (ii) let $[x, y]$ be the geodesic in X that connects x and y. Cover this interval by a finite number of translates of K. This determines a piecewise geodesic in K of the same length as the length of $[x, y]$. Hence this piecewise geodesic is a geodesic and is equal to $[x, y]$ by the uniqueness of geodesics. $\square$

The above proof appears in several different guises throughout the literature. Notice that it shows that a torsion free group acting geometrically on a uniquely geodesic metric space never has a strict fundamental domain. This is because torsion free groups have a quotient which has fundamental group isomorphic to that of the acting group. However any convex subset of a uniquely geodesic space is also uniquely geodesic and hence contractible.

2.2 The Davis Complex

One of the ways to think about this thesis is as giving a comparison between the Davis complex and an arbitrary CAT(0) space which an RACG acts. In this section we give the definition of the Davis complex. The actual definition will not play a substantial role here. However, the properties listed in Proposition 2.2.8 will be used mostly for comparison purposes.

2.2.1 Definitions

Before defining the Davis complex, we need to define a natural way of assigning a simplicial complex to any poset. Given any simplicial complex Δ, there exists a natural poset on the collection of all simplicies determined by inclusion. On the other hand, most posets do not come from such a construction. However, there is a natural simplicial complex determined by any poset. For any poset $\mathcal{P}$, we will now define the simplicial complex $||\mathcal{P}||$ by declaring the vertices of $||\mathcal{P}||$ to be the elements of the poset $\mathcal{P}$, and the simplicies of $||\mathcal{P}||$ to be the set of chains, $p_1 < \cdots < p_n$, of the poset $\mathcal{P}$.

Remark 2.2.1. If the poset $\mathcal{P}$ is the poset of faces of a simplicial complex ordered by inclusion, then $||\mathcal{P}||$ is the (face poset of) barycenter subdivision of of this simplicial complex.

Definition 2.2.2. Suppose that a special subgroup, W_T, is finite. Then, we call W_T a *spherical subgroup* and T a *spherical index set*.

For a RACS the notions of a flag and a spherical index set correspond.

Definition 2.2.3. Let $\mathcal{S}(W, S)$ be the set of all spherical index sets of a Coxeter group W. This collection forms a poset under inclusion. We define $\mathcal{K}(W, S)$ as $||\mathcal{S}(W, S)||$.

Definition 2.2.4 (The Davis Complex)**.** Let $W\mathcal{S}(W, S)$ be the collection of all spherical cosets, that is the set

$$\{ wW_T \mid T \in \mathcal{S}(W, S) \text{ and } w \in W \} .$$

This collection, like $\mathcal{K}(W, S)$, forms a poset under inclusion, and we define $\Sigma(W, S) := ||W\mathcal{S}(\mathcal{W}, \mathcal{S})||$. We will refer to $\Sigma(W, S)$ as the *Davis complex of* (W, S), or simply as the *Davis complex* when (W, S) is understood from context.

We first note that in the order stated above, $w_1 W_{T_1} \leq w_2 W_{T_2}$ if and only $W_{T_1} \subset W_{T_2}$ and $w_1 W_{T_2} = w_2 W_{T_2}$. Because of this there is a well defined group action $W \curvearrowright \Sigma(W, S)$ induced by $w \cdot w' W_T := ww' W_T$, and this action has a strict fundamental domain which can be identified with $\mathcal{K}(W, S)$ via $T_1 \subset \cdots \subset T_n \mapsto W_{T_1} \subset \cdots \subset W_{T_n}$.

2.2.2 Some Properties of the Davis Complex

Throughout this section, (W, S) is a fixed RACS, $\Sigma := \Sigma(W, S)$ and $K := \mathcal{K}(W, S)$. Since we will not make explicit use of the metric on the Davis complex, we merely say that the metric is defined so that unit cubes are glued in the obvious way.

Theorem 2.2.5 ([Mou88])**.** Σ *is* CAT(0). [1] $\square$

The following is proven in many places, (e.g in [Rad01]). We essentially copy the proof here to give the reader a better understanding of how the Davis complex and CAT(0) geometry can combine to a geometric proof of a highly non-trivial result.

Proposition 2.2.6. *In a Coxeter group, every finite subgroup is contained in a conjugate of a spherical subgroup.*

Proof. Let H be a finite subgroup of G. Then, by 1.3.14 there is a point x fixed by H. That is to say that $H \subseteq \mathrm{St}(x)$. Let $(x)^*$ be the smallest simplex of Σ which contains x. Then we have for all $h \in H$ that $h \cdot (x)^* \cap (x)^*$ is non-empty (it contains x) and is a simplex (the action of W on Σ is simplicial). Thus, by minimality, we must have that H actually preserves $(x)^*$.

As a simplex of Σ, $(x)^*$ has the form $w_1 W_{T_1} \subset \cdots \subset w_n W_{T_n}$. Since $|T_1|$ is

[1] This was proved originally in the case that W is a RACG in [Gro87].

strictly less than $|T_i|$ for all other i between 1 and n we have $hw_1W_{T_1} = w_1W_{T_1}$ for all $h \in H$. That is $H \subseteq w_1W_{T_1}w_1^{-1}$. $\square$

The only way the above will be used is in the guise of the following corollary which follows immediately from the proposition and the fact that spherical subgroups of (W, S) are direct products of cyclic groups of order 2.

Corollary 2.2.7. *If the order of a non-identity element of a* RACG *is finite, then the order is two.* $\square$

From the point of view taken here, the following properties are more important and could be taken as axioms which define the Davis complex. Moreover, they are exactly the properties that we wish to realize in more general situations. They are stated in such a way that the correspondence will become clear in later sections.

Proposition 2.2.8. *For any i, $j \in I$, the following hold in Σ:*

D1 *The action of W on Σ is geometric.*

D2 *Σ is a* UGM *space.*

D3 $\mathcal{T}_i = \{\mathrm{T}_i^y, \mathrm{T}_i^n\}$.

D4 *If $j \in \mathsf{c}(i)$, then $s_j \cdot \mathrm{T}_i^y \neq \mathrm{T}_i^y$ if and only $j = i$.*

D5 *If $j \in \mathsf{kl}(i)$, then $s_j \cdot \mathrm{T}_i^y \subset \mathrm{T}_j^y$.*

Proof. The action is cocompact since K is compact (it is a finite simplicial complex). It also is not hard to see that the action is proper. The half spaces are the sets

$$\mathrm{T}_i^y := (W^{s_i}\cdot K) \setminus \Sigma^{s_i} \quad \text{and} \quad \mathrm{T}_i^n := s_i \cdot \mathrm{T}_i^y.$$

The set T_i^y corresponds to the elements of w who cannot start with s_i and the set T_i^n corresponds to the elements of W who can start with s_i. All properties follow easily from these definitions, see [Dav08]. For instance, $i \in \urcorner(w)$ if and only if $i \in \urcorner(s_j w)$ for $j \in \mathsf{lk}(i)$. $\square$

The reader should note that these are equivalent to the four properties listed in the introduction.

Definition 2.2.9. We say that a quadruple (W, S, X, ρ) is *equivalent to the Davis complex* if it satisfies the conditions of Proposition 2.2.8.

We make this definition because we find it clearer to use an axiom-like approach to the Davis complex. There are no situations in the remainder of this document where we discuss the Davis complex or use any other properties than the ones listed above. For instance, the proposition below is provable from the definition of the sets T_i^y and T_i^n given above, but we wish to focus on the geometric significance of the above proposition.

For instance,

Proposition 2.2.10. *If (W, X, S, ρ) is equivalent to the Davis complex, then ρ is properly nested.*

Proof. If not, then for some flag, $\mathfrak{f}$, $X^{\mathfrak{f}} \subset X^{w(\mathfrak{f})}$. That is there is a point fixed by $w(\mathfrak{f})$ but not fixed by s_i for some $i \in \mathfrak{f}$. This means that $w(\mathfrak{f}) \cdot \mathrm{T}_i^y = \mathrm{T}_i^y$. On the other hand, the various parts of 2.2.8 tell us that it is an RACP. □

For a quadruple equivalent to the Davis complex, a corollary of The Main Theorem (see page 49) will tell us that the set

$$\bigcap_{i \in I} (\mathrm{T}_i^y \cup X^{s_i})$$

is a rigid strict fundamental domain for the action. This statement in itself can be extracted from lecture notes given by Bestivina, [Bes]. (This statement is true by definition in the Davis complex itself, this is one of the reasons that we deal with spaces equivalent to the Davis complex instead of the Davis complex itself).

Part II

Main Constructions

Chapter 3

Right-Angled Coxeter Pair Basics

Throughout this part, we assume that (W, S) is a RACS. Also unless otherwise stated, we assume that $W \overset{\rho}{\curvearrowright} X$ is a geometric action of W on a non-empty CAT(0) space X (by an action ρ). Also, we write $I = I(S)$. We call the tuple (W, S, X, ρ) a *pre-right-angled Coxeter pairing (pre-*RACP*)*. We will define a right-angled Coxeter pairing in the first section.

3.1 The Goal

Let us start with what will be referred to as *the basic lemma* and its corollaries.

Lemma 3.1.1 (Basic Lemma)**.** *For any r_1 and r_2 in W, $X^{r_1} \cap X^{r_2} \neq \emptyset$ if and only if $r_1 \circ r_2$ and $r_1^2 = r_2^2 = 1$.*

Proof. If $X^{r_1} \cap X^{r_2} \neq \emptyset$, then $X^{r_1 r_2} \neq \emptyset$. So, r_1, r_2 and $r_1 r_2$ are elliptic isometries and as such have finite order (by properness of the action). Since we are in a RACG, this implies that that order is two, so the elements commute. Conversely, if r_1 and r_2 commute, then $W_{\{1,2\}}$ is finite and Lemma 1.3.15 says that the fixed point set is non-empty. $\square$

Corollary 3.1.2. *The following are equivalent:*

(i) $X^{s_i} \cdot w \cap X^{s_i} \neq \emptyset$*,*

(ii) $w \in W_{\mathsf{c}(i)}$*,*

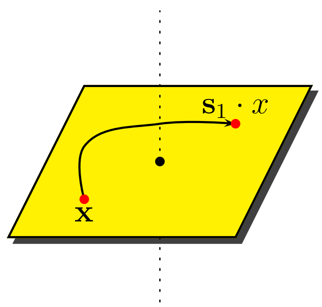
(iii) $X^{s_i} \cdot w = X^{s_i}$*.*

Proof. All that needs to be shown here is that (i) implies (ii). By the lemma, we know $w s_i w^{-1}$ commutes with s_i. The result now follows from Corollary 1.2.4 (in the case that $i = j$). $\square$

We slightly generalize the notations in Definition 1.3.12.

Definition 3.1.3. Let $Y \subseteq X$ be connected. For $w \in W$, define $\mathcal{T}_w(Y)$ to be the set of components of $X \setminus X^w$. (For $i \in I$, we write $\mathcal{T}_{s_i}(Y)$ as $\mathcal{T}_i(Y)$ or as $\mathcal{T}_i$ when $Y = X$).

Note that since all elements of $W_{\mathsf{c}(i)}$ preserve X^{s_i}, $W_{\mathsf{c}(i)}$ acts on $\mathcal{T}_i$. Note also that there is no reason to expect that $|\mathcal{T}_i| > 1$. For example, if we have a $\mathbb{Z}_2$ acting on the square via a 180 degree rotation then we have that $|\mathcal{T}_1| = 1$.



We now introduce a condition which roughly says that nowhere does the action look like this 180 degree rotation.

Definition 3.1.4. Suppose that $w \in W$ preserves Y. We say that w *strongly separates* Y if w acts freely on $\mathcal{T}_w(Y)$.

Notice that

Proposition 3.1.5. *For any $w \in W$ the following are equivalent.*

(*i*) *X^w meets every connected w-invariant set.*

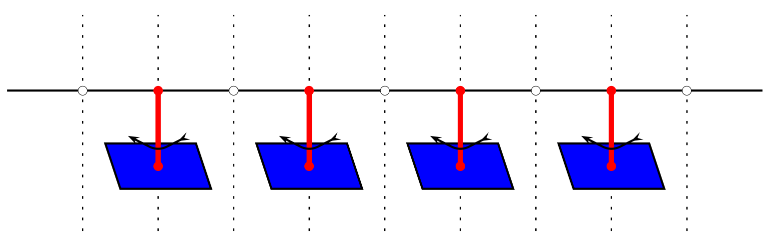
(*ii*) *X^w separates every connected w-invariant set that is not contained in X^w.*

(*iii*) *For every $x \in X$, and for every path γ connecting x and $w{\cdot}x$, we have that $\gamma \cap X^w \neq \emptyset$.*

(*iv*) *w strongly separates X.*

Proof. First, we point out that all conditions imply that $X^w \neq \emptyset$ and so $w^2 = 1$. Now, $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iv)$ follow from definition. To see that $(i) \Rightarrow (iii)$ notice that for any path γ from x to $w{\cdot}x$, $w{\cdot}\gamma \cup \gamma$ is a connected w-invariant set. Also, $(i) \Rightarrow (iv)$ since any w invariant $T \in \mathcal{T}_w$ is connected and does not meet X^w. Next, $(iv) \Rightarrow (iii)$ since (iv) implies that x and $w{\cdot}x$ are in different components of $X \setminus X^w$. Lastly, $(iii) \Rightarrow (ii)$ since if (ii) did not hold for some w-invariant set C, then x and $w{\cdot}x$ would be in the same path component of $X \setminus X^w$. $\square$

Also, note that strong separation is also slightly stronger than the requirement that X^w separates X in the usual sense (hence the adverb *strongly*). For instance



shows an action by the infinite dihedral with one generator acting by a generalized reflection while the other one does not. However both generators have fixed point sets which separate the space. On the other hand the requirement is weaker than requiring w act via a reflection, *i.e.* we do not require that w separates the space into two convex components.

Reflections $\rightarrow$ Strongly Separates $\rightarrow$ Separates

The following follows from definitions and the basic lemma.

Lemma 3.1.6. *If* $X^{s_i} \subseteq X^{s_i s_j}$ *then there exists a basic transvection* $\mathrm{Tv}_{i,j}$. $\square$

Thus for the rest of the document we maintain the assumption that $X^{s_i} \not\subseteq X^{s_i s_j}$.

We now present the main definition of this thesis.

Definition 3.1.7. A pre-right-angled Coxeter pairing (W, S, X, ρ) is a *right-angled Coxeter pairing* (RACP) if after a transvection the following holds for all $i \in I$:

> *Given any i-flag,* $\mathfrak{f}$, *and* $T \in \mathcal{T}_i$ *if* $w(\mathfrak{f}) \cdot T = T$, *then* $w(\mathfrak{f})$ *strongly separates* T.

This implies that every element of S strongly separates X; for, by definition, $X^{s_i} \cap T = \emptyset$ for every $T \in \mathcal{T}_i$.

For an RACP (pre-RACP), when any of W, S, X, or ρ are clear or irrelevant, we will omit mention them and also call what remains an RACP (pre-RACP). For instance the quadruple $(W, S, \Sigma(W, S), \phi_D)$ containing the Davis complex will simply be denoted $\Sigma(W, S)$. As such, we will also call the Davis complex itself an RACP. We will also carry over adjectives which describe the group, the generating set, the space, or the action associated to a particular RACP to describe the RACP as a whole. For instance, we say that two RACP's are equivalent if their corresponding RACS are equivalent. Also, we will say that an RACP has a strict fundamental domain whenever the associated action has a strict fundamental domain.

Examples 3.1.8.

1. 2.2.8 tells us that the Davis complex of any given RACS is an RACP.
2. If $W \curvearrowright T$ is a geometric action and T is a tree, then (W, T) is an RACP. This is because every path in a tree between two points contains the geodesic between those two points. (However, in this case the main theorem below is a consequence of Bass-Serre theory, see [Ser89] [the Bass-Serre tree of a any group generated by torsion is a strict fundamental domain]).

Our main theorem — to be proved in two parts, one in Section 6.1 and the other in Section 6.3 — is

The Main Theorem. *Every* RACP *has a strict fundamental domain.*

Remark 3.1.9. Unfortunately Theorem 3.1 does not give a necessary condition for an action to have a strict fundamental domain. However, as we proceed it will be clear exactly how to weaken the conditions above to be equivalent to the existence of a strict fundamental domain for $W \curvearrowright X$.

Before giving the proof, we will spend a lot of time building up preliminary results.

3.2 Basic Notions I

In this section we discuss some of the basic notions that pertain to non-commutability. Let

$$\mathcal{T} := \bigcup_{i \in I} \mathcal{T}_i.$$

By Definition 3.1.7 of an RACP, $|\mathcal{T}_i| > 1$ (in the Davis complex and classical Coxeter group theory the number is exactly two) and s_i acts freely on T_i. Note that these sets may even be infinite (see Remark 4.1.7). Next let

$$\widetilde{\mathcal{T}}_i := \left\{ T \cup X^{s_i} \;\middle|\; T \in \mathcal{T}_i \right\}.$$

We will refer to elements of $\widetilde{\mathcal{T}}_i$ by putting tildes over them. That is if $T \in \mathcal{T}_i$, then $\widetilde{T} := T \cup X^{s_i}$. (In the Davis complex $\widetilde{T}$ is simply $\overline{T}$, the topological closure of T. However, this is certainly not the case in general).

Remark 3.2.1. When dealing with sets of sets we will abuse notation and denote the set of sets and its union in the identical manner. It should be clear from context what is meant. This will be particularly useful when talking about the orbit of a set under the action of a group. For instance, for $T \in \mathcal{T}_i$, $W_{\mathsf{lk}(i)} \cdot T$ will denote either a subset of X or a subset of $\mathcal{T}_i$.

Definitions 3.2.2. By Lemma 3.1.1 we know that if $i \bullet j$, then $X^{s_i} \cap X^{s_j} = \emptyset$. Moreover, since X^{s_j} is connected, there exists a unique component in $\mathcal{T}_i$ that contains X^{s_j}. We denote this component by T^y_{ij}.

We say that

$$\mathcal{B} = \{b_1, \ldots, b_m\} \subseteq \mathsf{kl}(i)$$

make up *distinct a-components* if $\mathrm{T}^y_{ab_i} \neq \mathrm{T}^y_{ab_j}$ for all $1 \leq i, j \leq m$. We say that $\mathcal{B}$ *exhausts the* `yes` *a-components* if $\mathcal{B}$ make up distinct a-components and all `yes` components are equal to $\mathrm{T}^y_{ab_i}$ for some $b_i \in \mathcal{B}$. In this case we use the notation $\underline{\mathrm{kl}}(i) := \mathcal{B}$. (we could also think of $\underline{\mathrm{kl}}(i)$ is as $\mathsf{kl}(i)/\sim$ where $j_1 \sim j_2$ if and only if $j_1, j_2 \in \mathsf{in}(T)$ for some $T \in \mathcal{T}_i$.)

We define T^n_{ij} as $s_i \cdot \mathrm{T}^y_{ij}$. We say that T^y_{ij} is a `yes` component (or more specifically as `yes` i-component), and we refer to T^n_{ij} as a `no` component (or more specifically as a `no` i-component). We set $\mathrm{T}_{ij} := \mathrm{T}^y_{ij} \cup \mathrm{T}^n_{ij}$. Furthermore, we define

$$\mathrm{T}_i^{\epsilon} := \bigcup_{j \in \mathsf{kl}(i)} \mathrm{T}_{ij}^{\epsilon} \quad {}^{1}\text{for } s_i \notin \mathfrak{Z}(W) \text{ and } \quad \mathrm{T}_i^{\epsilon} := X \quad \text{otherwise.}$$

Finally, we define

$$\mathrm{T}^y := \bigcap_{i \in I} \mathrm{T}_i^y.$$

We make parallel definitions of all using the sets in $\widetilde{\mathfrak{T}}_i$. Thus, for instance,

$$\widetilde{\mathrm{T}^y} := \bigcap_{i \in I} \bigcup_{j \in \underline{\mathrm{kl}}(i)} \left(\mathrm{T}_{ij}^y \cup X^{s_i} \right).$$

For any of these sets T, if any combination of W, X, S or ρ are not clear from context, we indicate this by $\mathrm{T}(L) := \mathrm{T}$ where L is this combination. In particular $\widetilde{\mathrm{T}^y}$ depends on all four.

In a perfect world the set $\widetilde{\mathrm{T}^y}$ will always be a strict fundamental domain. However there are many reasons why this set will not be the case. Much of what will be done in the remaining sections will be focused on changing the action ρ to an action with the same orbits and the generating set S to an equivalent generating set so that $\widetilde{\mathrm{T}^y}$ is as close as possible to a strict fundamental domain.

Remark 3.2.3. If W has a non-trivial center (so that $W = W' \times W_{\mathfrak{z}(W)}$) we have that $\widetilde{\mathrm{T}^y}(W, X) = \widetilde{\mathrm{T}^y}(W', X)$. This is because central elements do not have any `yes` components to contribute to $\widetilde{\mathrm{T}^y}(W, X)$. In this situation however the

[1]Throughout this document, ϵ is to be interperted as being either y or n.

set $\widetilde{\mathrm{T}^y}(W, X)$ is clearly not a strict fundamental domain whenever the action is faithful. Later, when we have a STUFed RACP we will use the finite central subgroup to cut $\widetilde{\mathrm{T}^y}$ up into pieces.

In many cases when W is f-irreducible, the set $\widetilde{\mathrm{T}^y}$ will turn out to be the strict fundamental domain proposed in Theorem 3.1. Indeed, we will see this to be the case whenever X has the geodesic extension property (even if W is f-reducible). Another example where this always happens is in the Davis complex. (As we just pointed out, the set $\widetilde{\mathrm{T}^y}$ is *never* a strict strict set when the action of $\mathfrak{Z}(W)$ on X is non-trivial[2].)

The following will be a consequence of the proof of the main theorem.

Theorem 3.2.4. *Suppose that W has an outer automorphism group consisting of only graph symmetries (that is any two generating sets are conjugate) and that for all $i \in I$, $\mathcal{T}_i$ contains no bounded components (this happens if X has the geodesic property). Then for any generating set S, $\widetilde{\mathrm{T}^y}(W, S, X)$ is a strict fundamental domain for the action of W on X. Moreover in this case, $\mathcal{T}_i$ consists of exactly two elements for every $i \in I$.* $\square$

Roughly, this theorem says that the more that we need to change $\widetilde{\mathrm{T}^y}(W, S)$ for a generic action, the more complicated the outer automorphism group of W becomes.

[2]The action of the center is always trivial whenever a RACG acts on a space with the geodesic extension property

Remark 3.2.5. We may discuss T^y_{ij} without explicitly saying that $i \bullet j$.

3.3 Basic Notions II

In this section we talk about some basic notions pertaining to commutability. If $i \circ j$, then $s_i \cdot X^{s_j} = X^{s_j}$ and $s_j \cdot X^{s_i} = X^{s_i}$. Thus, we have an induced action $W_{\mathsf{c}(i)} \curvearrowright \mathcal{T}_i$. In the Davis complex, $|\mathcal{T}_i| = 2$, and $W_{\mathsf{lk}(i)} \curvearrowright \mathcal{T}_i$ is trivial. However, as we can see from most of of the examples in the example section, nothing can be further from the truth in an arbitrary RACP.

The following comes directly from the definition. However, it is sufficiently important and sufficiently awkward to merit it being written as a lemma. If the reader does not believe this immediately, then he or she should take a second to think through this with some of the examples in the example appendix.

Lemma 3.3.1. *Suppose that $T \in \mathcal{T}_i$ and $j \circ \{i, k\}$. Then,*

(i) *$s_j \cdot T = T$ if and only $X^{s_i} \cap T \neq \emptyset$,*

(ii) *If $s_j \cdot T \neq T$, then there exists $T' \in \mathcal{T}_j$ so that $T \subseteq T'$,*

(iii) *and $s_j \cdot \mathrm{T}^y_{ik} = \mathrm{T}^y_{ik}$.*

Proof. Since T and $s_j \cdot T \in \mathcal{T}_i$, we have one direction of (i), and by definition of an RACP we have the other. Moreover, ii is a restatement of (i), as well as (iii) (when coupled with the basic lemma). $\square$

Corollary 3.3.2. *If* $s_j \cdot \mathrm{T}^y_{ik} \neq \mathrm{T}^y_{ik}$*, then* $j \bullet k$ *and* $\mathrm{T}^y_{ik} \subseteq \mathrm{T}^y_{jk}$.

Proof. By (i) above, we know that $X^{s_j} \cap X^{s_k} \subseteq X^{s_j} \cap \mathrm{T}^y_{ik} = \emptyset$. Therefore, by the basic lemma, $j \bullet k$. Now, by (ii) above, there exists $T \in \mathfrak{T}_j$ with $X^{s_k} \subseteq \mathrm{T}^y_{ij} \subseteq T$. Thus, $X^{s_k} \cap T \neq \emptyset$ and so $T = \mathrm{T}^y_{jk}$. $\square$

3.4 The Part of `no` That We Don't Understand

The following section covers the notion of a wedge pair (and more generally a wedge tuple). Roughly, wedge pairs come about whenever $X^{s_i} = X^{s_j}$ for $i \neq j$ (the word *roughly* is used because there is no way to tell whether the various fix point sets are equal by the action alone, however when they are equal the phenomena discussed in this section does indeed occur). By the way that the Davis complex is constructed, this type of phenomena never occurs there. However, this is something that we will have to deal with in this more general setting. We first run through a quick example.

Example 3.4.1. The right half of Figure 3.1 is understood to be the indicated portion of the regular tree of valence four. We describe the action of 3 generators – s_1, s_2, and s_3– on this space in Table 3.1 and its caption.

	1	2	3	4	5	6	7
s_1	2	1	5	4	4	–	–
s_2	5	3	2	4	1	–	–
s_3	–	–	–	6	5	4	–

Table 3.1: The table details the action of the example on Figure 3.1. The "–" mean that the destination vertex does not appear on the portion of $\mathbb{T}_4$ depicted in the figure.

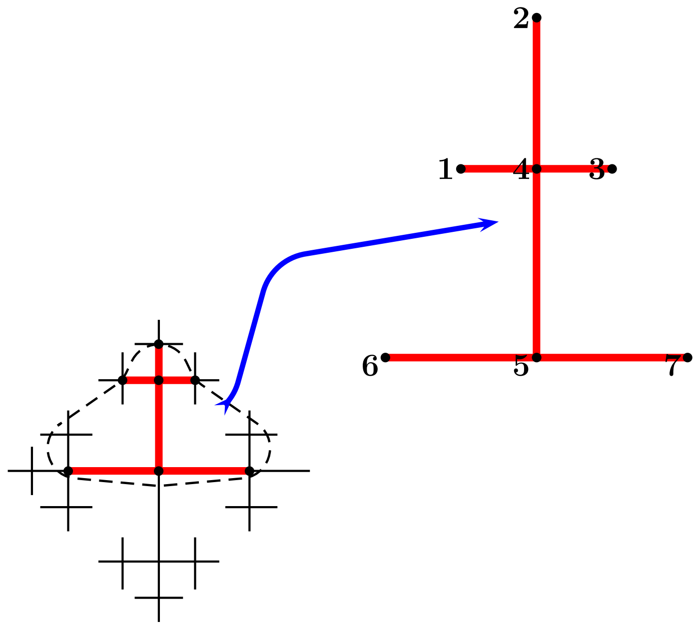


Figure 3.1: A wedge pair corresponding to Example 3.4.1

Notice that both X^{s_1} and X^{s_2} divide the space into 4 components. However, for each there is only one corresponding `yes` component — T^y_{13} and T^y_{23}. Moreover, $\mathrm{T}^y_{13} = \mathrm{T}^y_{23}$ and $\mathrm{T}^n_{13} \cap \mathrm{T}^n_{23} = \emptyset$. Lastly, we note that the component $s_2 \cdot \mathrm{T}^y_{13} = s_1 \cdot \mathrm{T}^y_{23}$ is not a `yes` nor a `no` component for neither s_1 or s_2.

Proposition 3.4.2 (The Misunderstood Ones)**.** *The following are equivalent:*

(i) $s_2 \cdot \mathrm{T}^y_{1a} \neq \mathrm{T}^y_{1a}$ *and* $s_1 \cdot \mathrm{T}^y_{2a} \neq \mathrm{T}^y_{2a}$

(ii) $T^y_{1a} = T^y_{2a}$

(iii) $s_2 \cdot T^n_{1a} = s_1 \cdot T^n_{2a}$

(iv) $s_2 \cdot T^y_{1a} \neq T^y_{1a}$ *and* $s_2 \cdot T^n_{1a} \cap T^n_{2a} = \emptyset$

(v) $s_2 \cdot T^y_{1a} \neq T^y_{1a}$ *and* $s_2 \cdot T^n_{1a} \not\subseteq T^n_{2a}$

Proof. $(i) \Rightarrow (ii)$ follows at once from Corollary 3.3.2, while $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v)$ hold by definition.

$(iii) \Rightarrow (i)$ We suppose that $s_1 \cdot T^y_{2a} = T^y_{2a}$. By definition, $s_1 \cdot T^y_{2a} = T^y_{2a}$ if and only if $s_1 \cdot T^n_{2a} = T^n_{2a}$. Now if $s_1 \cdot T^n_{2a} = T^n_{2a}$, then by (iii) we must have that $s_2 \cdot T^n_{1a} = T^n_{2a}$, *i.e.*, $T^n_{1a} = T^y_{2a}$. Now by definition, $X^{s_a} \subset T^y_{2a}$ and thus $X^{s_a} \subset T^n_{1a}$. But this implies that s_1 stabilizes T^y_{1a}, which is impossible by the definition of an RACP.

$(ii) \Rightarrow (iv)$ If $T^y_{1a} = T^y_{2a}$, and $s_2 \cdot T^n_{1a} \cap T^n_{2a} = \emptyset$, then $T^n_{1a} \cap T^y_{1a} = T^n_{1a} \cap T^y_{2a} \neq \emptyset$. But of course this is impossible since $T^y_{1a} \cap T^n_{1a} = \emptyset$.

$(v) \Rightarrow (i)$ Suppose that $s_1 \cdot T^y_{2a} = T^y_{2a}$ and $s_2 \cdot T^y_{1a} \neq T^y_{1a}$. Then by Lemma 3.3.1 on page 53, we must have that $T^y_{1a} \subset T^y_{2a}$. But this says that

$$s_2 \cdot T^n_{1a} \subseteq s_2 s_1 \cdot T^y_{2a} = s_2 \cdot T^y_{2a} = T^n_{2a}. \qquad \square$$

Definition 3.4.3. If the pair of `no`-components, $\{T^n_{1a}, T^n_{2a}\}$, satisfy any of the conditions above, we call the pair an *a-wedge pair*. We call a set of pairwise wedge (a-)pairs a wedge (a-)tuple. Note that an a-wedge tuple is uniquely determined by its flag. Because of this, we will define

$$\mathrm{T}^y_{\mathfrak{f}a} := \mathrm{T}^y_{ia} \text{ where } i \text{ is any element of } \mathfrak{f}. \tag{3.1}$$

and we call $\mathfrak{f}$ a *wedge a-flag*.

Part two of the above proposition immediately gives:

Corollary 3.4.4. *For* $\mathfrak{no}$*-components, T_1 and T_2, the relation*

$$T_1 \sim T_2 \textit{ if and only } \{T_1, T_2\} \textit{ is a wedge pair}$$

is an equivalence relation. □

We are now in position to make an important definition:

Definitions 3.4.5. Given a k-wedge tuple over a flag $\mathfrak{f}$, we define

$$\bigwedge_{i\in\mathfrak{f}} \mathrm{T}^n_{ik} := w(\mathfrak{f})\cdot\mathrm{T}^y_{jk}$$

where j can be taken as any element of $\mathfrak{f}$. Alternatively, building on the notation in (3.1)

$$\mathrm{T}^n_{\mathfrak{f}k} := w(\mathfrak{f})\cdot\mathrm{T}^y_{\mathfrak{f}k}.$$

Notice that in the latter notation

$$\mathrm{T}^n_{\{i\}k} = \mathrm{T}^n_{ik}.$$

We also make the convention that T^n_{ij} is a wedge 1-tuple. If this wedge tuple is a maximal wedge-tuple (meaning if s is flag to $\mathfrak{f}$, then $s \in \mathfrak{f}$), we define the collection of all

$$\left\{ \mathrm{T}^n_{\mathfrak{g}m} \mid \mathfrak{g} \subseteq \mathfrak{f} \right\}$$

an m-wedge wreath over the maximal flag $\mathfrak{f}$.

Chapter 4

STUFing: *S*tretching, *TU*rning, and *F*olding

There are many differences between the Davis complex, Σ, and an arbitrary RACP, X. For the purpose of finding a strict fundamental domain we show that some of the differences can be completely ignored while others need to be investigated further in order to understand their affect on the overall equivariant structure of the RACP.

In the first section of this chapter, we discuss many of the problems that we may encounter. In later sections of this chapter, we discuss apparent pathologies that can be massaged which shows that they (the pathologies) do not hinder us in our quest to find a strict fundamental domain. In those sections, we describe the turning and folding processes (collectively, known as the STUFing process) and will how they make an arbitrary RACP closely resemble the Davis complex. Moreover, these processes will be done in such a way that the existence of a strict fundamental domain in the STUF*ed space* will be equivalent to the existence of a strict fundamental domain in the original space. Moreover, after finding the strict fundamental domain in the STUFed RACP, it will be clear how to pass it to a strict fundamental domain in the original RACP.

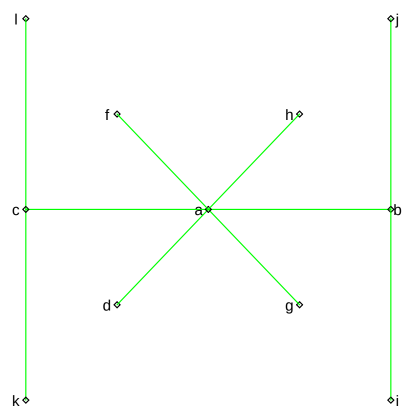


Figure 4.1: We take the RACG acting on the above tree with $I = \{1, 2\}$ with s_1 acting as the permutation $(a)(b,c)(d)(f)(g,h)(i,k)(j,l)$ and s_2 acting by the permutation $(a)(b,c)(d,f)(g)(h)(i,l)(j,k)$. Here $X^{s_1 s_2}$ is the subtree determined by the vertices $\{c, a, b\}$, X^{s_1} is determined by the vertices $\{a, f\}$, and X^{s_2} is determined by the vertices $\{a, g\}$. Notice that the RACP is not properly nested and not turned.

4.1 Overview

In this section we list some all of the differences between an RACP and the Davis complex that we will need to understand. The first three problems will in some sense be fixed while the latter problems listed in this section we will simply have to deal with. While we state all of the definitions here for a pre-RACP, we will not be able to accomplish anything unless we have an RACP.

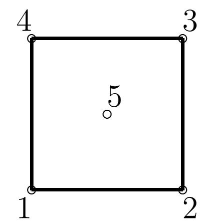
Definition 4.1.1. A pre-RACP is STUFed if it is turned (Definition 4.1.3), and folded (Definition 4.1.6) (STUF is an acronym for these two constituents plus stretching which was omitted in the final stages of the thesis writing, see below).

The point of this chapter is to prove:

Theorem. *Given any* RACP*,* (W, S, X, ρ)*, there corresponds an* RACP*,* $(W, S', X, \widehat{\rho}')$ *which is* STUF*ed and has a strict fundamental domain if and only if* (W, S, X, ρ) *has a strict fundamental domain.*

4.1.1 Stretching

In the Davis complex, every component of the complement of a fixed point sets is convex. Without this property arguments become slightly more cumbersome. For an example of space where the components are not convex consider the space we get from developing the strict fundamental domain consisting of the square with fixed point sets the four edges and the center. The group has the following diagram:



and a component of the space minus the center point is a plane with a point removed.

An easier example (without the geodesic extension property) is depicted in Figure 4.2. This space comes by first gluing together 3 squares end-to-end and then taking two copies of the three squares and gluing them together along the middle square along a free edge. Here just $\mathbb{Z}_2$ acts on this space and neither component of the space minus the fix point set is convex.

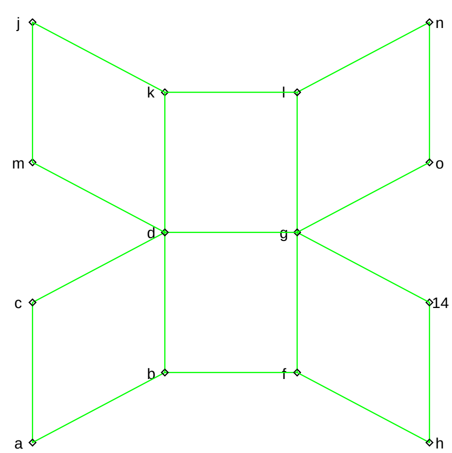


Figure 4.2: In the image components are not convex (the action is taken as $\mathbb{Z}_2$ flipping over the line $[d, g]$). (The space above consist of the indicated edges as well as the faces they enclose.)

Definition 4.1.2. A pre-RACP is *stretched* if T is a convex subset of X for every $i \in I$ and $T \in \mathfrak{T}_i$.

In the original version of this thesis the author went through some effort to change the RACP so that the new RACP became stretched. However, he has changed this particular aspect so that this is no longer necessary. The advantage of this approach lies in the fact that it uses fewer CAT(0) axioms. Also, the new argument is slightly more elegant. The downside however, without stretched components the argument in the folding section has become slightly more involved. Due to the last minute change in the thesis, the author keeps the same acronym "STUF" despite the fact that he no longer does the stretching procedure (which corresponded to the 'S' in STUF).

4.1.2 Turning

Recall that the Davis complex is properly nested. However, in general CAT(0) spaces all we know is that $X^{\mathfrak{f}} \subseteq X^{w(\mathfrak{f})}$ (even with the added assumption that X^{st} does not contain X^s for all $s \in S$ that was made before the definition of an RACP). For example, consider the tree in figure 4.1 with the action of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as described there. The fix point sets of s_1, s_2 and $s_1 s_2$ are each distinct. This is reflected in our strict fundamental domain which can be taken as the subtree determined by the vertices a, b, h, f, and j. In general we will need to redefine the way that our RACG acts on our space when this happens. This redefining process is known as *Turning*. We give a definition in terms of the action:

Definition 4.1.3. A pre-RACP is *turned* if for every i-flag w and for any $T \in \mathfrak{T}_i$, we have

$$w \cdot T \neq T.$$

Notice this is precisely the condition that makes of an RACP vacuous. Thus any turned pre-RACP is obviously an RACP.

We might as well prove a small proposition here.

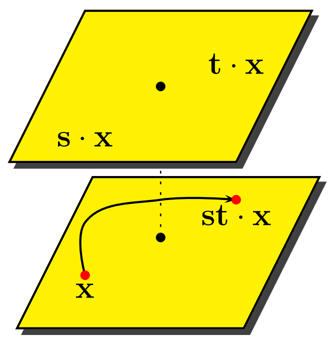
Proposition 4.1.4. *A pre-*RACP *is turned if and only if it is a properly nested* RACP*.*

Proof. We first show that a turned pre-RACP is properly nested. For, if not, there exists a flag, $\mathfrak{f}$, with $X^{w(\mathfrak{f})}$ not contained in X^{s_i} for some $i \in \mathfrak{f}$. This means that $X^{w(\mathfrak{f})}$ meets some i-component T (every point not in X^{s_i} lies in some $T \in \mathfrak{T}_i$) and hence $w(\mathfrak{f})$ fixes T. Thus the space is not turned. On the other hand, if (W, S) is not turned, then (by definition) there exists an $i \in \mathfrak{f} \subseteq I$ and $T \in \mathfrak{T}_i$ ($\mathfrak{f}$ a flag) with

$$w(\mathfrak{f}) \cdot T = T.$$

Thus, by definition of an RACP, we know that $X^{w(\mathfrak{f})} \cap T \neq \emptyset$. But, we have that $X^{s_i} \cap T = \emptyset$. So, in particular $X^{w(\mathfrak{f})} \not\subseteq X^{s_i}$. And therefore, (W, S, X, ρ) is not properly nested. $\square$

Remark 4.1.5. On the other hand, a pre-RACP can be properly nested and not turned (and hence not an RACP). For instance if we glue two squares together at a point and have $\mathbb{Z}_2 \times \mathbb{Z}_2$ act via having one generator permute the two squares while having the other generator also permute the two squares but also apply a 180 degree twist, then the product acts by individually rotating each square by 180 degrees. This action is not turned since this product fixes the components of each of the two generators. It is not an RACP because while this product fixes each component, it does not separate either (strongly or otherwise).



(The two squares are glued together along the dotted line)

This proposition will often be used without explicit mention and was stated here to help build the readers intuition. However, in a majority of the proofs that we will encounter here, the original definition will be more useful than the equivalent one stated in the proposition.

4.1.3 Folding

The following question is pressing and basic: How do we know that we have chosen our generators correctly? Often, the way that the generating set S acts on X is by no means optimal. Even in simple examples, the fix point sets of S do not have a configuration which leads to any natural way to attack the strict fundamental domain problem. For instance, the fixed point sets can be far enough apart so that $\widetilde{\mathrm{T}^y}(S)$ is not even compact. This means that it cannot be contained in a strict fundamental domain for the RACP (which must be compact by cocompactness). The process that we will describe to pick out an optimal generating set is called *Folding*. It will depend on the fact that the turning procedure has already been completed. We now state a formal definition. As with turning, this definition is given in terms of the actions $W_{\mathsf{c}(i)} \curvearrowright \mathcal{T}_i$.

Definition 4.1.6. A pre-RACP is *folded* if for every $i, k, l \in I$ and $w \in W_{\mathsf{c}(i)}$, we have that

$$w \cdot \mathrm{T}^y_{ik} = \mathrm{T}^y_{il} \quad \text{implies that} \quad \mathrm{T}^y_{ik} = \mathrm{T}^y_{il}.$$

Theorem 4.3.9 in Section 4.3 states that:

Theorem. *Given any* RACP (W, S, X, ρ) *there exists a partial conjugation* φ *so that* (W, S, X, ρ) *is folded.*

While the definitions pertaining to folding make sense for any pre-RACP, we can only establish this theorem for RACP's.

4.1.4 Number of Components

The first difference that one should notice between the Davis complex and an arbitrary RACP is the number of components that the complement of the fixed point set of a generator can have (that is the number of elements of $\mathcal{T}_i$ for various i). In the Davis complex this number is exactly two. However in our more general setting it is even possible that the fixed point set of some generator separates the space into an infinite number of components. This can happen even in one-ended spaces [1]. Unlike with the three previous differences, we will not "fix" this problem.

Remark 4.1.7. In this remark we show how easy it is to have an RACP with $|\mathcal{T}_i| = \infty$. This is one of the inevitable remarks where the author says a lot and verifies nothing. Suppose that (W, S) has a non-trivial partial conjugation, $\mathrm{Pc}_{i,T}$. Let $S' = \mathsf{c}(i) \cup T$ and $S'' = \mathsf{c}(i) \cup S \setminus T$. Take $\mathrm{K}(W_{S'}, S')$ and $\mathrm{K}(W_{S''}, S'')$ the associated strict fundamental domains and glue them together along $\mathrm{K}(W_{\mathsf{c}(i)}, \mathsf{c}(i))$ to get a set K which has the induced mirror structure. Now consider the obvious semi-direct product extension of W generated by W and $\mathrm{Pc}_{i,T}$. This is a RACG with generators given by $(S \setminus \{s_i\}) \cup \{\mathrm{Pc}_{i,T}, s_i\mathrm{Pc}_{i,T}\}$ which has an obvious geometric action on $\mathcal{U}(K, W)$ (for a definition of $\mathcal{U}(K, W)$

[1] A CAT(0) space is 1-ended if the removal of any compact set from X leaves X connected, see [BH99]

see [Dav08]) with the property that the complement of the fixed point set of $\mathrm{Pc}_{i,T}$ has an infinite number of components. Roughly, the reader can think of this process as coming about from "hiding" a partial conjugation.

4.1.5 Bounded Components

In the Davis complex, since X^s separates X into exactly two components which are interchanged by the isometry s, all components share the same boundedness properties. For instance, if we have the usual RACG action on the plane, and we equivariantly glue a flat strip to a fix point set of one of the generators, that generator will separate the resulting space into four components. Two of which are a bounded Hausdorff distance away from the fix point set and the other two are not. This is more of an annoyance than anything. However, worse than this, the components of various fix point sets in X^s (when viewed as a CAT(0) space in its own right) will have varying boundedness properties even if we force our original space to not have these problems (see Example A.6 below). We will discuss what to do with this in Corollary 5.

4.1.6 Wedge Pairs

We have already discussed *wedge pairs* in Section 3.4. These wedge pairs never occur in the Davis complex since the action of $W_{\mathsf{lk}(i)}$ on $\mathcal{T}_i$ is always trivial (this is the statement of Proposition 2.2.8). However, based on Proposition 3.4.2, these wedge pairs will not cause any great difficulty as long as we are careful.

4.2 The Turning Procedure

Recall that the purpose of *turning* is to fix problems that arise when $X^{s_i s_j}$ becomes too large relative to X^{s_i} and X^{s_j}, e.g., $X^{s_i s_j} \not\subseteq X^{s_i} \cap X^{s_j}$. Of course, when s_i and s_j do not commute, $X^{s_i s_j}$ is empty, and we have nothing to worry about. In this section we alter our RACP so that the problem that existed in the original RACP is erased. We will see that the change is purely superficial. The superficiality of this process is indicated by the fact that the orbit of every point x is not changed after turning. Since the existence of a strict fundamental domain is defined solely in terms or the orbit of a point we will be able to prove the main theorem for a turned RACP.

In this section we will construct a new action $\rho' \colon W \to \mathrm{Isom}(X)$ so that (W, S, X, ρ') is a *turned* RACP. In this section, our space X, and our RACS

(W, S) will remain fixed. As such, we will reference an RACP only by its action ρ.

The following lemma makes reference to only one action, so we use the usual notation $\rho(s)(x) = s \cdot x$.

Lemma 4.2.1. *Suppose $T \in \mathfrak{T}_i$ and $w \in W$. If $w \cdot T \in \mathfrak{T}_i$, then $w \in W_{\mathsf{c}(i)}$.*

Proof. Suppose that $w \cdot T = T'$ for $T' \in \mathfrak{T}_i$. Then

$$w \cdot \overline{T} = \overline{T'} \quad \text{and} \quad w \cdot \left(\overline{T} \setminus T\right) = \overline{T'} \setminus T'.$$

But it is clear that $\overline{T} \setminus T$ and $\overline{T'} \setminus T'$ are nonempty and are contained in X^{s_i} (as $\widetilde{T}$ and $\widetilde{T'}$ are closed). Thus Corollary 3.1.2 applies to give the result. □

We now begin to describe the turning procedure. We start out with the assumption that the offending word is $s_i s_j$, a word length equal to 2. Let $T \in \mathfrak{T}_i$ be so that $s_i s_j \cdot T = T$. This is equivalent to saying that $\mathrm{X}_\rho^{s_i s_j}$ meets T but $\mathrm{X}_\rho^{s_j}$ does not. We now define the new action ρ' piecewise as follows:

$$\rho'(s_j)(x) = \begin{cases} \rho(w s_i s_j w^{-1})(x) & \text{if } w \in W_{\mathsf{c}(j)} \text{ and } x \in \rho(w)T; \\ \rho(s_j)(x) & \text{otherwise.} \end{cases} \tag{4.1}$$

And for all other $k \neq j$, we define $\rho'(s_k) = \rho(s_k)$. We call this the *turned action*. One way to think about this action is as the action one would get after applying transvections, but only locally. Indeed in the paragraph below Lemma 3.1.6 we discussed what to do if $X^{s_j} \subseteq X^{s_i s_j}$ globally.

Remark 4.2.2. Our goal in 4.1 was to define a new action with $X^{s_i s_j} \subseteq X^{s_j}$. So, for certain, on $\widetilde{T} \cup s_j{\cdot}\widetilde{T}$ we needed to have the fix point sets of s_j and $s_i s_j$ reversed. However, since $W_{\mathsf{c}(j)}$ preserves X^{s_j}, this would also require that for all $T' \in W^{\rho'}_{\mathsf{c}(j)}{\cdot}T$, that $s_j{\cdot}T' = T'$. This is why (4.1) was defined the way that it was.

Proposition 4.2.3. *Given any* RACP*, ρ, there is an* RACP*, $\hat{\rho}$, which is turned. Moreover, a set is a strict fundamental domain for ρ if and only if it is also a strict fundamental domain for $\hat{\rho}$.*

Proof. We will show that ρ' as defined in (4.1) is closer to being turned then ρ. In order to do this, there are several things to verify about the map ρ'. Throughout this, we will often revert to writing $\rho(w)(x)$ as $w{\cdot}x$, but we always be pedantic about our reference to $\rho'(w)$.

The function ρ' is well defined. In addition to $T \in \mathfrak{T}_i$, as assumed, we will show that $T \in \mathfrak{T}_j$. That is T is a component of $X \setminus X^{s_j}$. First note that Lemma 3.3.1 says there exists a $T' \in \mathfrak{T}_j$ with $T \subseteq T'$. If we can show $s_i{\cdot}T' \neq T'$, then a second application of the same lemma would show that $T = T'$. But this is easy since $s_i s_j{\cdot}T = T$ implies $s_i s_j{\cdot}T' = T'$. If $s_i{\cdot}T' = T'$, then we would have $s_i s_j{\cdot}T' = T' = s_j{\cdot}T' = T'$. But, we cannot have $s_j{\cdot}T' = T'$

by definition of an RACP. Thus we also cannot have $s_i{\cdot}T' = T'$.

Suppose that w_1, $w_2 \in W_{\mathsf{c}(j)}$. Then, $w_1{\cdot}T \cap w_2{\cdot}T \neq \emptyset$, implies that we have that $w_2^{-1}w_1 \cdot T = T$ (since $W_{\mathsf{c}(j)} \curvearrowright \mathcal{T}_i$). Hence by Lemma 4.2.1 just above, $w \in W_{\mathsf{c}(i)}$. Thus for $x \in w_1{\cdot}T$,

$$\left(w_1 s_i s_j w_1^{-1}\right){\cdot}x = \left(w_2 s_i s_j w_2^{-1}\right){\cdot}x$$

which shows that ρ' is well defined.

The function defines an action. If $x \notin w{\cdot}T$ for any $w \in W_{\mathsf{c}(j)}$, then the map is clearly an action at x. Moreover, it is clear that for all $s \in S$ (even for $s_j = s$ where $\rho'(s_j)$ leaves invariant $w{\cdot}T$) that $\rho'(s)^2 = 1$. Thus, we need only show that $\rho'(s_j s_k)^2(x) = x$ for $k \in \mathsf{lk}(j)$ and $x \in w{\cdot}T$. (This argument will not depend on the fact that $w \in W_{\mathsf{c}(j)}$). In this situation:

$$\begin{aligned}
\rho'(s_k s_j)^2(x) &= \rho'(s_k)\rho'(s_j)\rho'(s_k)\rho'(s_j)(x) && [\rho' \text{ defined on generators}]\\
&= \rho'(s_k)\rho'(s_j)\rho(s_k)(\left(w s_i s_j w^{-1}\right){\cdot}x) && [\text{still in } w{\cdot}T]\\
&= \rho'(s_k)\rho'(s_j)(\left(s_k w s_i s_j w^{-1}\right){\cdot}x) && [\text{right term in } s_k w{\cdot}T]\\
&= \rho'(s_k)((s_k w) s_i s_j (s_k w)^{-1}(s_k w s_i s_j w^{-1}){\cdot}x) && [(4.1)]\\
&= \rho'(s_k)(s_k{\cdot}x) && [\text{simple cancelation}]\\
&= s_k s_k{\cdot}x && [k \neq j]\\
&= x.
\end{aligned}$$

This shows that we have an action.

Both actions have the same orbits. We will show that $W^{\rho}\cdot x = W^{\rho'}\cdot x$ for all $x \in X$. For this part, unlike the previous, we explicitly use ρ.

The inclusion $W^{\rho'}\cdot x \subseteq W^{\rho}\cdot x$ is clear. Indeed, this follows from the fact that elements of W were defined to act via ρ' in a way that they already did under ρ.

For the other inclusion we consider x and $\rho(w)(x)$ for some $x \in X$. We are looking for a $\bar{w} \in W$ with $\rho'(\bar{w})(x) = \rho(w)(x)$. Our proof will be by induction on $\ell(w)$.

Claim. We can assume that $x \in T$, $\rho(w)(x) \in W^{\rho}_{\mathsf{c}(j)}\cdot T$.

We claim that the only case that we need to consider is when x and $\rho(w)(x)$ are in $W^{\rho}_{\mathsf{c}(j)}\cdot T$ and $j \in \urcorner(w)$. Indeed, let $\omega = s_m \cdots s_1$ be a word representing w. If for each $i \leq m$ we had $\rho(s_i \cdots s_1)(x) \notin W^{\rho}_{\mathsf{c}(j)}\cdot T$, then we could define $w = w'$ and since $\rho(w) = \rho'(w)$ we would be done. So by replacing x by the first point in the orbit of x appearing in the list

$$(x, \rho'(s_1)(x), \ldots, \rho'(w)(x))$$

that lives in $W^{\rho}_{\mathsf{c}(j)}\cdot T$, we assume that $x \in \rho(w_1)T$ for $w_1 \in W_{\mathsf{c}(j)}$. Similarly, working from left to right in the word ω we can use induction to assume that

$\rho(w)(x) \in W^{\rho}_{\mathsf{c}(j)} \cdot T$. Thus we assume from now on that $\rho(w)(x) \in \rho(w_3)T$ for some $w_3 \in W_{\mathsf{c}(j)}$.

Next, we can actually assume that $w_1 = 1$ since in this case $W^{\rho'} \cdot x = W^{\rho'} \cdot x'$ for some element of $x' \in T$ because $\rho(w_1) = \rho'(w_1)$ if $j \notin \urcorner(w_1)$ and $\rho(w_1) = \rho(w_1 s_i)$ if it is. ■

Now onto our induction. In the case that $w \in S$ we define $\bar{w} = w$ if $w \neq s_j$ and $\bar{w} = s_i s_j$ if $w = s_j$. Our assumption that $x \in T$ shows that we have taken care of the $\ell(w) = 1$ case.

We assume the result for $\ell(w) = m-1$. If $j \notin \urcorner(w)$ and s is, then $\ell(sw) = \ell(w) - 1$ and therefore $\rho(sw)(x) = \rho'(w_2)(x)$ by the inductive hypothesis and $\rho(w)(x) = \rho'(s)\rho'(w_2)(x)$. Moreover, if $\rho(s_j w) \notin W^{\rho}_{\mathsf{c}(j)} \cdot T$, then $\rho(w)(x) = \rho'(s_j w_2)(x)$. We can therefore assume that $j \in \urcorner(w)$.

Finally, we we define $\bar{w} = w_3 s_i s_j w_3^{-1} w_2$ and we show that this $\bar{w}$ has the desired properties.

If $j \notin ⅂(w_3)$, then $\rho(w_3) = \rho'(w_3)$ so

$$
\begin{aligned}
\rho'(\bar{w})(x) &= \rho'(w_3 s_i s_j w_3^{-1} w_2)(x) \\
&= \rho(w_3)\rho'(s_i s_j)\rho'(w_3^{-1} w_2)(x) \\
&= \rho(w_3)\rho'(s_i s_j)\rho(w_3^{-1} s_j w)(x) \\
&= \rho(w_3)\rho'(s_i s_j)\rho(s_j w_3^{-1} w)(x) \\
&= \rho(w_3)\rho(s_j)\rho(s_j w_3^{-1} w)(x) \qquad\qquad \text{(since } \rho(s_j w_3^{-1} w)(x) \in \rho(s_j)T\text{)} \\
&= \rho(w_3 s_j s_j w_3^{-1} w)(x) \\
&= \rho(w_3 w_3^{-1} w)(x) \\
&= \rho(w)(x).
\end{aligned}
$$

Alternatively, if $j \in ⅂(w_3)$ then $j \notin \mathcal{L}(s_j w_3)$ and so $\rho'(w_3 s_j) = \rho(w_3 s_j)$. Moreover

$$\begin{aligned}
\rho'(\bar{w})(x) &= \rho'(w_3 s_i s_j w_3^{-1} w_2)(x) \\
&= \rho'(w_3 s_j s_i s_j s_j w_3^{-1} w_2)(x) \\
&= \rho(w_3 s_j)\rho'(s_i s_j)\rho'(s_j w_3^{-1} w_2)(x) \\
&= \rho(w_3 s_j)\rho'(s_i s_j)\rho(s_j w_3^{-1} s_j w)(x) \\
&= \rho(w_3 s_j)\rho'(s_i s_j)\rho(w_3^{-1} w)(x) \\
&= \rho(w_3 s_j)\rho(s_j)\rho(w_3^{-1} w)(x) && \text{(since } \rho(w_3^{-1} w)(x) \in T) \\
&= \rho(w_3 w_3^{-1} w)(x) \\
&= \rho(w)(x).
\end{aligned}$$

Therefore, we have the desired result.

The new action is also a geometric action. It is clear that the action is proper and cocompact since we can "do" exactly the same things with our new action as we did with our old one. That is for every $x \in X$,

$$W^{\rho} \cdot x = W^{\rho'} \cdot x$$

; *i.e.*, the orbit of x in the original action is the same as the orbit of x in the new action as we have just shown. Lemma 1.3.20, with

$$\mathcal{C} = \{\, X^{s_j} \,\}$$

shows us that the new action is also by isometries.

We still have an RACP. This is the case since in our original definition, Definition 3.1.7, we insisted that bad pairs $(\mathfrak{f}, T)$ had corresponding group elements which strongly separates T. Notice that this is the only place in this document that we use the fact that minimal elements separate as well as generators.

Remark 4.2.4. With hindsight it is clear that the definition of an RACP was specifically engineered so that the paragraph makes sense. However, we can now see what I mean in Remark 3.1.9 since there is no reason to require that the original s_j strongly separate the space since it was going to be locally replaced by $s_i s_j$ anyway.

This process works. Notice that $\mathrm{X}_{\rho}^{s_j} \subset \mathrm{X}_{\rho'}^{s_j}$. This is because in the original action, the fix point of s_j does not enter T. However, it does in the new action. On the other hand, we show

Claim. The fixed point set of $s_i s_j$ is strictly smaller under the action of ρ' then it is under the action of ρ.

Proof of Claim. We simply show that $\mathrm{X}_{\rho'}^{s_i s_j} \cap W_{\mathsf{c}(j)} T = \emptyset$. This will show strict inclusion since $\mathrm{X}_{\rho'}^{s_i s_j} \cap T \neq \emptyset$ and the actions correspond outside of $W_{\mathsf{c}(j)}^{\rho} \cdot T$. Now, suppose that $\rho'(s_i s_j)(x) = x$ for some $x \in \rho(w) T$ and $w \in W_{\mathsf{c}(j)}$. First, let us see what happens to T:

$$\rho'(s_i s_j)\rho(w)T = \rho(s_i)\rho(w s_i s_j w^{-1})\rho(w)T = \rho(s_i w)\rho(s_i s_j)T = \rho(s_i w)T.$$

Thus, since $x \in \rho(w)T$ and $\rho(s_i s_j)(x) = x$, we must have that

$$\rho(w)T \cap \rho(s_i w)T \neq \emptyset.$$

And as in the proof that ρ' was well defined, we have that $w s_i w^{-1} \in W_{\mathsf{c}(i)}$ and $\rho(s_i w)T = \rho(w)T$. Now, by Corollary 1.2.4 (in the case that $i = j$), we have that $w \in W_{\mathsf{c}(i)}$. Hence, $\rho(s_i)T = T$, which contradicts the definition of an RACP. $\blacksquare$

Higher products. The procedure is similar. If $(w(\mathfrak{f}), k)$ are a bad pair we pick $m \in \mathfrak{f}$ to play the role of s_j in everything above, and we substitute $w(\mathfrak{f})s_m = w(\mathfrak{f} \setminus m)$ everywhere we saw s_i.

Existence of a strict fundamental domain. By the above, the orbit of any point in the original action and the turned action are the same. Therefore, a set is a strict fundamental domain for ρ if and only if it is also a strict fundamental domain for ρ'. $\square$

Remark 4.2.5. It is fairly clear from the Cayley graph (or the Davis complex) and the fact that the properties of quasi-isometries (see nearly any paper or book on geometric group theory for relevant definitions, for instance [BH99]) that X^{st} cannot escape from X^s by too much. However, we will see this as an obvious corollary of results to come. We will make no use of this fact, but it may help your intuition when it comes to this kind of construction.

4.3 The Folding Procedure

In this section, in proofs by induction, we will use a right action instead of the left actions. This is only because $T{\cdot}(s_1 \cdots s_m)$ is more appeasing for inductive purposes than $(s_m \cdots s_1){\cdot}T$.

4.3.1 Pre-folded and Intermediately Folded RACP's

In this section, we will need to assume that our RACP is turned. As already pointed out in Section 4.1.3, when we are given an arbitrary RACP, there is no reason to expect that the fix point sets of the generators have any sort of decent configuration. See Example A.3 on page 128 in the example section for some typical problems that may arise. (As a matter of fact, we must worry about folding whenever we have basis conjugating automorphisms that are not inner automorphisms). Here are the relevant definitions for this chapter. (The definition of a folded RACP was previously given in Definition 4.1.6).

Definitions 4.3.1. An RACP is *folded* if for every $i, k, l \in I$ and $w \in W_{\mathsf{c}(i)}$ we have that

$$\mathrm{T}^y_{ik} \cdot w = \mathrm{T}^y_{il} \quad \text{implies} \quad \mathrm{T}^y_{ik} = \mathrm{T}^y_{il}, \tag{4.2}$$

and we call an RACP *intermediately folded* if (4.2) holds whenever w is an i-flag. We call an RACP *pre-folded* if for all k-wedge tuples over i-flags $\mathfrak{f}$

$$\mathrm{T}^n_{\mathfrak{f}k} \neq \mathrm{T}^y_{il}$$

for all $k, l \in I$

One should think of an RACP as being folded whenever the action of the various centralizers do not mix up distinct `yes` components, an RACP as being intermediately folded whenever the action of flags do not mix up distinct `yes` components, and pre-folded when the action of flags do no mix up specific kinds of `yes` components.

Folded $\rightarrow$ Intermediately Folded $\rightarrow$ Pre-Folded

As such a folded space is intermediately folded, and an intermediately folded space is pre-folded. The plan of attack is to first show that each of these implications can be turned around. Lastly, we will show how any RACP differs from a pre-folded RACP by a partial conjugation. Since this clearly does not effect the question of existence of a strict fundamental domain it will allow us to assume for the remainder of the thesis that every RACP is folded.

The first step is the most difficult. We will state it now before coming back to prove it a short while.

Lemma 4.3.2. *An intermediately folded space is folded.*

Before proving this, we cover a lemma (for our lemma, yes I am allowed to sin heavily in my thesis) that will be useful in this context, and in many other places as well.

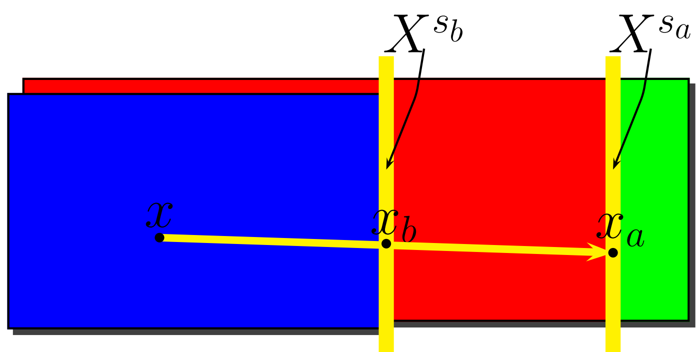


Figure 4.3: This figure corresponds to Lemma 4.3.3(i). The red region in the figure is the a component T, and the blue region is the b component T'. Notice in the image that $T = \mathrm{T}^y_{ab}$

Lemma 4.3.3. *Suppose $a \bullet b$, $T \in \mathfrak{T}_a$ and $T' \in \mathfrak{T}_b$.*

(i) *If $T \cap T' \neq \emptyset$, then either $T = \mathrm{T}^y_{ab}$ or $T' = \mathrm{T}^y_{ba}$.*

(ii) *If $T \neq \mathrm{T}^y_{ab}$, then $T \cdot s_b \subset \mathrm{T}^n_{ba}$.*

Proof. We first tackle (i). Let x be in the intersection. We assume that $T' \neq \mathrm{T}^y_{ba}$ and show that $T \neq \mathrm{T}^y_{ab}$. Figure 4.3 tells much of the story, but we write the argument down in words. The assumption that $T' \neq \mathrm{T}^y_{ba}$ means x and X^{s_a} lie in different b-components, i.e., the geodesic from x to some (hence any) point $x_a \in X^{s_a}$ must meet X^b. Call a point in this intersection x_b. Since $[\,x_a, x\,]$ and X^{s_a} are convex, the geodesic from x_a to x cannot meet X^{s_a} again after exiting it. In particular, $[\,x_b, x\,]$ does not meet X^{s_a}. But this is precisely the condition that $x \in \mathrm{T}^y_{ab}$.

For (ii), we note that if $T_a \neq \mathrm{T}^y_{ab}$, then $T_a \subset \mathrm{T}^y_{ba}$. This is because any geodesic from X^{s_b} to any point of $X \setminus \mathrm{T}^y_{ab}$ must pass through X^{s_a}. Hence by definition, $T \cdot s_b \subset \mathrm{T}^n_{ba}$. $\square$

Proof of 4.3.2. We suppose that our space is not folded, and we consider the set of all quadruples (g, a, b, c) $(g \in W,\ a \bullet \{b, c\})$ which satisfy the following:

- $g \in W_{\mathsf{c}(a)}$,
- $\{b, c\}$ make up distinct a-components, and
- $\mathrm{T}^y_{ab} \cdot g = \mathrm{T}^y_{ac}$.

In this case we say that the quadruple is a *bad quadruple*. We define the *length of the bad quadruple* as the word length of the corresponding group element. The set of bad quadruples is not empty since we are assuming that (W, X) is not folded. Let (w, a, b, c) be a bad quadruple with minimal length. We will first show that w is a flag, then we show there exists $\widetilde{a} \in I$ with w

an $\widetilde{a}$-flag, and that $(w, \widetilde{a}, b, c)$ is a bad quadruple showing that (W, X) is not intermediately folded.

Our starting point is the equation

$$\mathrm{T}_{ab}^{y} \cdot w = \mathrm{T}_{ac}^{y}. \tag{4.3}$$

Our plan is to come at (4.3) from two sides by using a wide barrage of previously established techniques including Corollary 3.3.2, both parts of Lemma 4.3.3, the fact that (W, X) is turned, and the results in Section 1.2.2 pertaining to flagged centralizer decompositions.

For starters, we denote the flag decomposition of w by $w(\mathfrak{f}_1)\cdots w(\mathfrak{f}_r)$. Our goal is to show that $r = 1$, which establishes the fact that w is a flag (not necessarily and a-flag). We assume that $r > 1$, and we bring $w(\mathfrak{f}_r)$ over to the right hand side in (4.3) which yields:

$$\mathrm{T}_{ab}^{y} \cdot (w(\mathfrak{f}_1)\cdots w(\mathfrak{f}_{r-1})) = \mathrm{T}_{ac}^{y} \cdot w(\mathfrak{f}_r). \tag{4.4}$$

For $1 \le i \le r$, let a_i be the first letter of the word representing $w(\mathfrak{f}_i)$ in a corresponding flagged centralizer decomposition of w. Recall that part of the definition of a flagged centralizer decomposition required that $a_i \bullet a_{i+1}$.

If $\mathrm{T}_{ab}^{y} \cdot s_{a_1} = \mathrm{T}_{ab}^{y}$, then $s_{a_1} w$ would have shorter word length and $(s_{a_1} w, a, b, c)$

would still be a bad quadruple, contradicting the minimality of (w, a, b, c). Thus, we must have that $\mathrm{T}^y_{ab}\cdot s_{a_1} \neq \mathrm{T}^y_{ab}$. Similarly, $\mathrm{T}^y_{ac}\cdot s_{a_r} \neq \mathrm{T}^y_{ac}$. Thus, Corollary 3.3.2 gives,

$$\mathrm{T}^y_{ab} \subseteq \mathrm{T}^y_{a_1 b} \quad \text{and} \quad \mathrm{T}^y_{ac} \subseteq \mathrm{T}^y_{a_r c}. \tag{4.5}$$

Moving up to the next level on the left hand side of (4.4), we now claim that

$$\mathrm{T}^y_{a_1 b}\cdot w(\mathfrak{f}_1) \neq \mathrm{T}^y_{a_1 a_2} \tag{4.6}$$

as well. To show this, we first assume that b and a_2 make up the same a_1-components. Then, since our RACP is turned, (4.6) holds in this case. On the other hand, if if b and a_2 make up distinct a_1-components, then (4.6) is a consequence of the minimality of w (since $(w(\mathfrak{f}_1), a_1, b, a_2)$ would be a shorter bad quadruple). Thus, in either case, the claim, (4.6), holds. Therefore, Lemma 4.3.3(ii) applies and shows that

$$\mathrm{T}^y_{a_1 b}\cdot w(\mathfrak{f}_1) \subseteq \mathrm{T}^y_{a_2 a_1}.$$

Therefore, we have so far shown that

$$\mathrm{T}^y_{1b}\cdot \left(w_{\mathfrak{f}_1} w_{\mathfrak{f}_2} \cdots w_{\mathfrak{f}_{r-1}}\right) \subseteq \mathrm{T}^y_{a_2 a_1}\cdot \left(w_{\mathfrak{f}_2} \cdots w_{\mathfrak{f}_{r-1}}\right).$$

If $r \geq 3$, the identical logic as applied to get (4.6) also shows

$$\mathrm{T}^y_{1b}\cdot \left(w_{\mathfrak{f}_1} w_{\mathfrak{f}_2} w_{\mathfrak{f}_3} \cdots w_{\mathfrak{f}_{r-1}}\right) \subseteq \mathrm{T}^y_{a_3 a_2}\cdot \left(w_{\mathfrak{f}_3} \cdots w_{\mathfrak{f}_{r-1}}\right).$$

If we continue in this way and combine what we get with (4.5) we get:

$$\mathrm{T}^y_{a_{(r-1)}a_{(r-2)}}\cdot w_{\mathfrak{f}_{(r-1)}} \supseteq \ldots \supseteq \mathrm{T}^y_{ab}\cdot\left(w_{\mathfrak{f}_1}\cdots w_{\mathfrak{f}_{(r-1)}}\right) = \mathrm{T}^y_{ac}\cdot w_{\mathfrak{f}_r} \subseteq \mathrm{T}^y_{a_r c}\cdot w_{\mathfrak{f}_r}.$$

Hence, in particular we have that

$$\mathrm{T}^y_{a_{(r-1)}a_{(r-2)}}\cdot w_{\mathfrak{f}_{(r-1)}} \cap \mathrm{T}^y_{a_r c}\cdot w_{\mathfrak{f}_r} \neq \emptyset.$$

So Lemma 4.3.3(i) applies to show that either

$$\mathrm{T}^y_{a_{(r-1)}a_{(r-2)}}\cdot w_{\mathfrak{f}_{(r-1)}} = \mathrm{T}^y_{a_{(r-1)}a_r} \qquad \text{or} \qquad \mathrm{T}^y_{a_r c}\cdot w_{\mathfrak{f}_r} = \mathrm{T}^y_{a_r a_{(r-1)}}.$$

The two cases differ only in notation, so we assume that the second of the two statments just made. Since our RACP is turned, we must have that $a_{(r-1)}$ and c make up distinct a_r-components. Thus, we must have that $(w(\mathfrak{f}_r), a_r, a_{r-1}, c)$ is a bad quadruple and $r = 1$, which was as desired.

We now prove our second assertion about the existence of $\widetilde{a}$. For a flag group element w with $\mathrm{T}^y_{ab}\cdot w = \mathrm{T}^y_{ac}$ as in the rest of the proposition, we have for any $s \in \daleth(w)$ that

$$\mathrm{T}^y_{ab}\cdot w = \mathrm{T}^y_{ac} \tag{4.7}$$

implies that for $\overline{w} = ws$ that we have on the left hand side of (4.7) (recall that by convention, $s_* = s$.)

$$\mathrm{T}^y_{ab} \cdot w \subseteq \mathrm{T}^n_{*\,b} \cdot \overline{w} = \mathrm{T}^y_{*\,b} \cdot w$$

and

$$\mathrm{T}^y_{ac} \subseteq \mathrm{T}^y_{*\,c} \cdot w$$

on the right hand side. Hence

$$\mathrm{T}^y_{ac} \subseteq \mathrm{T}^y_{*\,b} \cdot w \cap \mathrm{T}^y_{*\,c}.$$

In particular the intersection is non-empty, and since they are components:

$$\mathrm{T}^y_{*\,b} = \mathrm{T}^y_{*\,c} \cdot w.$$

Since we are assuming that our RACP is turned, we have $\mathrm{T}^y_{*\,b} \neq \mathrm{T}^y_{*\,c}$, and the result follows. □

The following is a Corollary of the proof. It will be stated again later as a theorem. This is because it does not fit in with the rest of the material here but the proof is the same as that result above.

Corollary 4.3.4 (Super-turning)**.** *In a* STUF*ed* RACP*, under the action of* $W_{\mathsf{lk}(i)}$ *on* $\mathfrak{T}_i$*, the stabilizer of a* `yes` *component is a special subgroup of* $W_{\mathsf{lk}(i)}$.

Proof. Suppose that $w \cdot T = T$ where T is a yes i-component and $w = s_1 \cdots s_m$. If w is a flag, then $X^w \cap T \neq \emptyset$. Since our space is turned (hence properly nested), $X^w = X^{s_1} \cap \cdots \cap X^{s_m}$. Thus $s_i \cdot T = T$ for all $1 \leq i \leq m$. If w is not a flag and there exists an s_i with $s_i \cdot T \neq T$ $(1 \leq i \leq m)$, then there must be a $1 \leq j \leq m$ with $j \neq m$ and $s_j \cdot T \neq T$. We might as well assume that $i \leq j$. Then, with the word $s_i \cdots s_j$ we are in the same situation that we were in the first part of the above proposition (notice that $\mathrm{T}^y_{ab} \neq \mathrm{T}^y_{ac}$ was never used) and this showed that our spaced was not folded. □

Remark 4.3.5. This special subgroup can range from being all of $W_{\mathsf{lk}(i)}$, as in the Davis complex, to being trivial as in several of the examples in Appendix A.

Getting back to the subject at hand (namely showing there exists an equivalent folded RACP), the next order of business is to show that a pre-folded space is an intermediately folded space.

Proposition 4.3.6. *If an* RACP *is pre-folded, then it is intermediately folded.*

Proof. Assume that the RACP is not intermediately folded. Then, by definition, there is an i-flag $w(\mathfrak{f})$ and an a and b making up distinct i-components with $\mathrm{T}^y_{ia} \cdot w(\mathfrak{f}) = \mathrm{T}^y_{ib}$. As in Lemma 4.3.2 we call the quadruple $(\mathfrak{f}, i, a, b)$ a *bad quadruple*. We assume that $|\mathfrak{f}|$ is no greater than that of any other flag appearing in a bad quadruple. If $\ell(w) = 1$, we are done. Thus, we assume that $\ell(w) > 1$. Let $j \in \mathfrak{f}$. Then Lemma 3.3.1 says that

$$\mathrm{T}^y_{ja} \cdot w(\mathfrak{f}) = \mathrm{T}^y_{jb}.$$

And by turning, a and b also make up distinct j-components. By minimality of $|\mathfrak{f}|$, all $j \neq j' \in \mathfrak{f}$ we must have $\mathrm{T}^y_{ja} \cdot s_{j'} \neq \mathrm{T}^y_{jb}$. This means that

$$\{\, \mathrm{T}^n_{ia} \;\mid \text{for } 1 \leq i \leq m \,\}$$

is a wedge m-tuple. $\square$

4.3.2 The Folding Process

Recapping where we are right now, we have shown that if (W, X, S') is pre-folded, then it is also folded. We now describe a process that takes an arbitrary RACP, (W, X, S), and changes the generating set S into a generating set S' with (W, X, S') pre-folded. Throughout the section both W and X (and the action ρ) are to remain fixed. We will reference an RACP only by the set S. For example, we will say that S is not pre-folded whenever (W, S, X) is not pre-folded.

We suppose that S is not pre-folded. That is there exists a flag $\mathfrak{f}$, a $k_1, k_2 \in I(S)$

with $\{k_1, k_2\} \bullet \mathfrak{f}$ with k_1 and k_2 making up distinct $\mathfrak{f}$ `yes` components, and

$$\mathrm{T}^y_{\mathfrak{f}k_1} \cdot w(\mathfrak{f}) = \mathrm{T}^n_{\mathfrak{f}k_1} = \mathrm{T}^y_{\mathfrak{f}k_2}. \tag{4.8}$$

We call the triple $(\mathfrak{f}, k_1, k_2)$ a bad triple. We also pick a base-point, $x_o \in X$, in order to guarantee that the process that we describe here converges. (This is akin to modding out by the inner automorphisms in the full automorphism group of W). In (4.8) we insist that $x_o \notin \mathrm{T}^y_{\mathfrak{f}k_1}$. This is not really a restriction since $(\mathfrak{f}, k_1, k_2)$ is a bad triple if and only $(\mathfrak{f}, k_2, k_1)$ is a bad triple. (Since $w(\mathfrak{f})$ is a flag and as such $w(\mathfrak{f})^{-1} = w(\mathfrak{f})$.) We define a *folding move* as a map

$$S \xrightarrow{\varphi = \varphi(\mathfrak{f}, k_1, k_1)} S'$$

with[2]

$$\varphi(s) = \begin{cases} w_{\mathfrak{f}}\, s\, w_{\mathfrak{f}}^{-1} & \text{if } * \in \mathsf{in}(\mathrm{T}^y_{\mathfrak{f}k_1}); \\ s & \text{otherwise.} \end{cases} \tag{4.9}$$

Lemma 4.3.7. *A folding move induces a basic partial conjugation (hence an automorphism) of the underlying* RACG.

Proof. This follows since the definition of φ is is equivalent to (4.9) with $\mathsf{in}(\mathrm{T}^y_{\mathfrak{f}k_1})$ replaced by $\mathsf{mt}(\mathrm{T}^y_{\mathfrak{f}k_1})$. This is because if X^smeets $\mathrm{T}^y_{\mathfrak{f}k_1}$ but is not contained in it, then it must meet $\overline{\mathrm{T}^y_{\mathfrak{f}k_1}}$, and hence $X^{\mathfrak{f}}$ (implying that s is flag

[2]Recall that $w_{\mathfrak{f}}$ and $w(\mathfrak{f})$ are two notations for the same thing

to $\mathfrak{f}$). ☐

A *configuration of fixed point sets*is simply a subset $S' \subset W$ (labeled by the same index set I) which differs from S by a partial conjugation. We assign to every configuration, $S' = \{r_i\}_{i \in I}$, two numbers $\mathrm{D}(S')$ and $\mathrm{Y}(S')$ where

$$\mathrm{D}(\bar{w}) := \sum_{i,j \in I} d(X^{r_i}, X^{r_j}) \tag{4.10}$$

$$\mathrm{Y}(\bar{w}) := |\left\{ \mathrm{T}^{y}_{ij}(W, S', X, \rho) \mid i \in I, j \in \mathsf{kl}(i) \right\} \tag{4.11}$$

Thus D measures the size of the configuration and Y measures the total number of `yes` components in the configuration. We can now define a lexicographic order on the set of configurations by defining $\bar{S}' \prec \bar{S}''$ if and only $\mathrm{D}(S') < \mathrm{D}(S'')$ or $\mathrm{D}(S') = \mathrm{D}(S'')$ and $\mathrm{Y}(S') < \mathrm{Y}(S'')$.

Lemma 4.3.8. *If*

$$S_1 \xrightarrow{\varphi(\mathfrak{f}, k_1, k_2)} S_2,$$

then $S_2 \prec S_1$.

Proof. Throughout the proof $\varphi(\mathfrak{f}, k_1, k_1)$ will be denoted by φ. We first note that $\mathrm{D}(S_2) \leq \mathrm{D}(S_1)$ with equality holding if and only if all terms defining D are equal. That is

$$d\big(X^{\varphi(s_i)}, X^{\varphi(s_j)}\big) = d(X^{s_i}, X^{s_j}) \tag{4.12}$$

for all $i, j \in I$. Indeed, this is clear since the sets $\widetilde{\mathrm{T}}^{y}_{ij}$ are each convex. Thus we assume that (4.12) holds and draw a contradiction.

The construction of φ guarantees that the number of `yes` i-components decreases (for all $i \in \mathfrak{f}$). Indeed, k_1 and k_2 make up distinct `yes` i-components before the application of φ, and they make up the same `yes` i-components afterwards. Thus, the only way that Y, which measures the total number of `yes` components for *all* elements of I, does not decrease after the application of φ is if there is some other $j \in I$ which gains an additional `yes` component after the application of φ. That is there exists l_1 and l_2 which make up the same `yes` j-components before the application of φ, but they make up distinct `yes` j-components after the application of φ.

We finish the proof by making a series of contradictions.

Case 1. j commutes with an element of $\mathfrak{f}$.

Step 1: Case one guarantees that $j \circ \mathfrak{f}$. Let α be a geodesic from some point in $X^{s_{l_1}}$ to some point in $X^{\varphi(s_{l_1})}$. The assumption that these two sets live in distinct j components means that α must meet X^{s_j}. Since from the vantage point of a point in $X^{s_{l_1}}$ the application of φ is the same as acting by $w(\mathfrak{f})$, this geodesic must also meet $X^{w(\mathfrak{f})}$. If these points are the same for any choices of such points, then the basic lemma gives us the conclusion of this step. On the other hand, if $i \in \mathfrak{f}$ and commutes with j, then s_j must fix the i-component where the geodesic meets X^{s_j}. But all of these components are wedge components and thus j fixes a component of all elements of $\mathfrak{f}$. Thus, Lemma 4.2.1 gives us the desired result.

Step 2: $\{j\} \cup \mathfrak{f}$ forms a wedge tuple If not then s_j must fix $\mathrm{T}^y_{\mathfrak{f}k_1}$. This would mean that $\mathrm{T}^y_{\mathfrak{f}k_1}$ strictly contains $\mathrm{T}^y_{jl_1} = \mathrm{T}^y_{jl_1}$. But this means that φ does the same thing to both s_{l_1} and s_{l_2} and as such they cannot possibly make up distinct j-components after the application of φ since they do not make up distinct j-components before the application of φ.

Step 3: Draw the desired contradiction What we have established so far guarantees that φ does not stabilize s_{l_1} nor s_{l_2} and thus the number of `yes` j-components is actually reduced. (Since k_1, k_2, l_1 and l_2 make up distinct j components before the application of φ, but they make up the same `yes` j-component after.

Case 2. j does not commute with any element of $\mathfrak{f}$. This case is simpler, and it can actually occur if we did not have our assumption above D. Indeed we claim that this second case actually guarantees that D decreases. The assumptions guarantee that X^{s_j} is strictly contained in one of the wedge components, say $\mathrm{T}^y_{\mathfrak{f}k_2}$. We are also guaranteed that the application of φ has different effects only one of s_{l_1} or s_{l_2}, say s_{l_1}. This means that $X^{s_{l_1}}$ lies in $\mathrm{T}^y_{\mathfrak{f}k_1}$ and convexity of $\widetilde{\mathrm{T}^y_{\mathfrak{f}k_2}}$ tells us that X^{s_j} and $X^{\varphi(s_{l_1})}$ are strictly closer together than X^{s_j} and $X^{s_{l_1}}$ were. This contradicts (4.12) and thus D must decrease. $\square$

We are now ready for our victory lap.

Theorem 4.3.9. *Given any S, there exist an equivalent S' which is folded and differs from S by a sequence of partial conjugations.*

Proof. Local finiteness of the fixed point sets, that we are always folding towards x_o, and the previous two lemmas above tell us that a finite number of folding moves

$$S \to \ldots \to S'$$

takes us from S to a folded RACP S'. $\square$

Remark 4.3.10. In the original version of this thesis, when stretching was involved (see Section 4.1.1) there was no need for a lexicographic ordering, but only the order induced by D was enough.

4.4 STUFed pairs

In this section we collect some results pertaining to STUFed pre-RACP's. Many results presented here are only here because they are best gathered into one section such as this.

In this section, for the most part, (and the remainder of the document) we assume that (W, S, X, ρ) is STUFed. The exception occurs when we want to highlight a consequence of a particular feature of STUF.

We repeat Corollary 4.3.4 here in a slightly more specific form and dub it a theorem.

Theorem 4.4.1. *For every $i \in I$ and $k \in \mathsf{kl}(i)$, we have that*

$$\mathrm{St}\,\mathrm{T}^y_{ik} = W_{\mathsf{mt}(\mathrm{T}^y_{ik})\setminus\mathsf{in}(\mathrm{T}^y_{ik})}$$

$\square$

The following proposition gives us a closest approximation of the situation that exists in the Davis complex. It says that for every $i \in I$, we (roughly) can partition the set $\mathcal{T}_i$ in half, each half stabilized by $W_{\mathsf{lk}(i)}$ and that s_i swaps the two halves (compare with Proposition 2.2.8).

Definition 4.4.2. For an i-component T, we denote $W_{\mathsf{lk}(i)}{\cdot}T$ by $\mathcal{O}_i T$ or simply $\mathcal{O}T$ when i is understood.

Proposition 4.4.3. *Let $a \in I$ and suppose $\{b_1, \dots, b_m\} = \underline{\mathrm{kl}}(a)$. Then the following sets are mutually disjoint*

$$\mathcal{O}\mathrm{T}^y_{ab_1}, \mathcal{O}\mathrm{T}^n_{ab_1}, \dots, \mathcal{O}\mathrm{T}^y_{ab_m}, \mathcal{O}\mathrm{T}^n_{ab_m}.$$

Proof. We have that for all $1 \leq i \neq j \leq m$

$$\left(\mathcal{O}\mathrm{T}^n_{ab_i} \cup \mathcal{O}\mathrm{T}^y_{ab_i}\right) \bigcap \left(\mathcal{O}\mathrm{T}^n_{ab_j} \cup \mathcal{O}\mathrm{T}^y_{ab_j}\right) = \emptyset$$

by the definition of folding and

$$\mathcal{O}\mathrm{T}^n_{ab_i} \cap \mathcal{O}\mathrm{T}^y_{ab_i} = \emptyset$$

by super-turning (Corollary 4.3.4 or Theorem 4.4.1). $\square$

The following have been floating around in the proofs of several lemmas, but it is worth writing them down here. This will provide the first half of the proof that $\widetilde{\mathrm{T}^y}$ is a strict set (it actually shows that T^y is a strict set, but this is does not have much value as this set can even be empty, see Example A.5). Notice how the proofs now look incredibly similar to the proofs that one would give in the case that $\mathcal{T}_i$ consisted of just two components for every $i \in I$.

Corollary 4.4.4. *Suppose that W is f-irreducible. Then,* (*i*) *If* $x \in \mathrm{T}^y_1$ *and* $\omega = s_1 \cdots s_m$ *has a centralizer decomposition of syllable length* r, *then we have* $x \cdot \omega \in \mathcal{O}\mathrm{T}^n_{a_r}$.

(*ii*) *If* $x \in \mathrm{T}^y$ *and* $w \neq 1$, *then* $x \cdot w \notin \widetilde{\mathrm{T}^y}$.

(*iii*) *For any* $i \bullet j$, $\widetilde{\mathrm{T}^n_i} \cap \widetilde{\mathrm{T}^n_j} = \emptyset$.

Proof. (*i*) Our proof is by induction on $\ell_C(\omega)$. If $\ell_C(\omega) = 1$, then $\omega \in W_{\mathsf{c}(1)}$, $a_r = 1$, and by definition $x \in \mathcal{O}\mathrm{T}^n_1$. Thus we assume that the result holds whenever $\ell_C(\omega) = r - 1$. Now by definition of a centralizer decomposition, $a_r \notin \mathsf{c}(a_{r-1})$. By the induction hypothesis, we know $x \cdot \big(s_1 \cdots s_{a_{r-1}}\big) \in \mathcal{O}\mathrm{T}^n_{a_{r-1}}$ and by the proposition this implies $x \cdot \big(s_1 \cdots s_{a_{r-1}}\big) \notin \mathrm{T}^y_{a_{r-1}a_r}$. Hence by Lemma 4.3.3 on page 81, we have

$$x \cdot \big(s_1 \cdots s_{a_{r-1}}\big) \in \mathrm{T}^y_{a_r a_{r-1}} \quad \text{thus} \quad x \cdot (s_1 \cdots s_{a_r}) \in \mathrm{T}^n_{a_r a_{r-1}}.$$

Lastly, by definition of a centralizer decomposition, $s_{a_r+1} \cdots s_m \in W_{\mathsf{lk}(a_r)}$ and so $x \cdot (s_1 \cdots s_m) \in \mathcal{O}\mathrm{T}^n_{a_r}$ as desired.

(*ii*) Since $x \in \mathrm{T}^y$, we must have that the hypothesis of (i) holds for every $i \in I$. Hence, $x \cdot w \in \mathcal{O}\mathrm{T}^n_a$ for some $a \in I$ and therefore it is not in T^y_a by the proposition. Moreover, $x \cdot w$ is not in X^{s_a}. Therefore, $x \cdot w$ does not lie in $\widetilde{\mathrm{T}^y}$.

(*iii*) If $\mathrm{T}^n_{ia} \cap \mathrm{T}^n_{jb} \neq \emptyset$, then by 4.3.3.(ii) either $\mathrm{T}^n_{ia} = \mathrm{T}^y_{ij}$ or $\mathrm{T}^n_{jb} = \mathrm{T}^y_{ji}$. Both contradict the proposition □

Definition 4.4.5. Define $\mathcal{R}_i := \mathfrak{T}_i \setminus \mathcal{O}(\mathrm{T}_i)$ and $\mathfrak{X}^{s_i} := X^{s_i} \cup \mathcal{R}_i$. We refer to the sets $\mathfrak{X}^{s_i}$ as *extended fixed point sets.*

Notice that the set $\mathfrak{X}^{s_i}$ is really the set $\widetilde{\mathcal{R}_i}$, but we prefer the notation just established because it puts the focus on X^{s_i} instead of $\mathcal{R}_i$. For instance, we have an analog of the basic lemma:

Lemma 4.4.6 (compare Lemma 3.1.1)**.** *$\mathfrak{X}^{s_i} \cap \mathfrak{X}^{s_j} \neq \emptyset$ if and only if $i \circ j$.*

Proof. If $i \bullet j$, then $X^{s_i} \cap X^{s_j} = \emptyset$ (by the basic lemma, Lemma 3.1.1) and $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ (by Lemma 4.3.3(i)). On the other hand, if $i \circ j$, then the result follows from the fact that $X^{s_i} \cap X^{s_j} \neq \emptyset$. □

The sets $\mathcal{R}_i$ and $\mathfrak{X}^{s_i}$ will concern us in the next chapter. With help from the below corollary and cocompactness, we will show that $\mathcal{R}_i$ are the set of components that are who are a finite Hausdorff distance from X^{s_i} in Theorem 5.1.7. Notice that

$$X = \mathcal{R}_i \cup \mathcal{O}(\mathrm{T}_i) \cup X^{s_i}.$$

Along the same lines of $\mathfrak{X}^{s_i}$, for $T \in \mathfrak{T}_i$, we define $\mathcal{O}^\star(T)$ to be the set $\mathcal{O}(T) \cup \mathcal{R}_i$. Notice that Lemma 4.3.3(i) and the basic lemma combine to say that for $i \bullet j$,

$$\widetilde{\mathcal{O}^\star(\mathrm{T}_i^n)} \cap \widetilde{\mathcal{O}^\star(\mathrm{T}_j^n)} = \emptyset. \tag{4.13}$$

Proposition 4.4.7. *Let i, $j \in I$. Suppose that for $w \in W$ we have*

$$\mathcal{X}^{s_i}{\cdot}w \cap \mathcal{X}^{s_j} \neq \emptyset.$$

Then, w can be written geodesically as $w_1 w_2$ where $w_1 \in W_{\mathsf{c}(i)}$ and $w_2 \in W_{\mathsf{c}(j)}$.

For the proof of the proposition and its corollaries we let $\bar{x} \in \mathcal{X}^{s_i}{\cdot}w \cap \mathcal{X}^{s_j}$ and let $x = \bar{x}{\cdot}w^{-1}$.

proof of 4.4.7. Write w as $w_1 w_2 w_3$ where $w_1 \in W_{\mathsf{c}(i)}$, $w_3 \in W_{\mathsf{c}(j)}$, and $w_2 \in {}^{\mathsf{c}(i)}W^{\mathsf{c}(j)}$. We first point out that w_2 also satisfies the hypothesis of the proposition, so we rename w_2 as w. We must show that $w = 1$. For any $a \in \urcorner(w)$ we have that $x \in \mathrm{T}^{y}_{ai}$. Moreover, by Corollary 4.4.4(i), for any $b \in \urcorner(w^{-1})$ we have that $\bar{x} \in \mathcal{O}\mathrm{T}^{n}_{b}$. But since $w \in {}^{\mathsf{c}(i)}W^{\mathsf{c}(j)}$ we must have that $b \bullet j$, and this contradicts (4.13) since $\bar{x} \in \mathcal{X}^{s_j}$. $\square$

Corollary 4.4.8 (compare Corollary 3.1.2)**.** *The following are equivalent:*

(i) $\mathcal{X}^{s_i}{\cdot}w \cap \mathcal{X}^{s_i} \neq \emptyset$,

(ii) $w \in W_{\mathsf{c}(i)}$,

(iii) $\mathcal{X}^{s_i}{\cdot}w = \mathcal{X}^{s_i}$.

Proof. Only (i) $\Rightarrow$ (ii) does not follow from definitions. But the proposition gives this immediately. $\square$

Corollary 4.4.9. *Suppose that* w *satisfies the same hypothesis as the proposition and let* $w = w_1 w_2$ *be the guaranteed geodesic factorization. Then,* $x \cdot w_1 \in \mathfrak{X}^{s_i} \cap \mathfrak{X}^{s_j}$.

Proof. Since $\bar{x} \in \mathfrak{X}^{s_j}$ we have that $\bar{x} \cdot w_2^{-1} \in \mathfrak{X}^{s_j}$ as well. Also, $x \in \mathfrak{X}^{s_j}$ tells us that $x \cdot w_1 \in \mathfrak{X}^{s_j}$. But, $x \cdot w_1 = \bar{x} \cdot w_2^{-1}$, by definition. □

The above corollary says that in particular if $x \in \mathcal{R}_i$, then $\mathcal{R}_i \cap \mathcal{R}_j \neq \emptyset$. That is if $x \in \mathcal{R}_i$,then $\mathcal{R}_i \cap \mathcal{R}_j \neq \emptyset$. Also, it tells us if the condition of the proposition holds, then $i \circ j$.

Chapter 5

A closer look at $W_{\mathsf{c}(i)} \curvearrowright X^{s_i}$

In this chapter we take steps in order to set up an induction over various fixed point sets. In doing this RACP's which split as a product with a finite factor and with spaces X which have what are called bounded components (see definitions below) naturally arise. It is tempting to ignore these problems since if we assume that our group is f-irreducible and that X has the geodesic extension property, then our RACP does not have to directly deal with these rather annoying conditions. However, by using an induction argument requires that we not only understand the action of W on X but also the action of the links and centralizers on the various fix point sets. And it is easy to see that in these subspaces it is unreasonable to expect that any of these conditions hold even if they hold in the original space X. For instance, if we have nested fix point sets (that is $X^{s_1} \subseteq X^{s_2}$) then the action of s_2 on X^{s_1} is not faithful. See A.6 for an example where there is a fixed point set that does not have the geodesic extension property, but the original space does have the geodesic extension property.

5.1 Unbounded Components

Definition 5.1.1. Let $i \in I$. By an *i-bounded component*, we mean a component $T \in \mathcal{T}_i$ which is a finite Hausdorff distance from X^{s_i}. That is the component does not contain a geodesic ray which can be extended to meet X^{s_i}. If T is not i-bounded, then we say that T is an *unbounded* i-component. If i is clear from context, we will simply refer to bounded and unbounded components (this is not the usual meaning of bounded unless X^{s_i} is compact).

Notation 5.1.2. We denote the set of unbounded i-components $\mathcal{T}_i^U$, and we denote the set of bounded i-components $\mathcal{T}_i^B$.

Remark 5.1.3. By definition, $\mathcal{T}_i = \mathcal{T}_i^B \coprod \mathcal{T}_i^U$. Also, weak convexity shows that if X has the geodesic extension property, then $\mathcal{T}_i = \mathcal{T}_i^U$.

The next couple of lemmas follow slightly more easily if the entire scope of CAT(0) geometry is used. However, we will stick with our trend of using only the results mentioned in Section 1.3 to prove our results. In particular, we do not use the standard CAT(0) fact that there is a unique projection onto any convex set in a CAT(0) space (see [BH99] for more details). This is because the unique projection property holds for fixed point sets in any UGM space.

Lemma 5.1.4. *For $a \in I$ and $x \in X$ we have that*

$$d(x, X^{s_a}) = d(s_a \cdot x, X^{s_a}) = \frac{1}{2} d(x, s_a \cdot x).$$

Proof. Since s_a always fixes $[\, x, s_a \cdot x\,]$, the midpoint, M, is the only point in this geodesic which lies in X^{s_a}. Let m be a point in X^{s_a} closest to x.[1] That is m is a point with $d(x, m) = d(x, X^{s_a})$. It remains to show that $m = M$. Since M also lies in X^{s_a}, we have that $d(x, m) \leq d(x, M)$. Also, since s_a acts as an isometry fixing m and M, $d(s_a \cdot x, m) \leq d(s_a \cdot x, M)$. Thus $d(x, m) + d(m, s_a \cdot x) \leq d(x, s_a \cdot x)$. Hence m lies on a geodesic from x to $s_a \cdot x$, and by uniqueness of geodesics, $m = M$. $\square$

Lemma 5.1.5. *Suppose $a \bullet b$, $x_a \in X^{s_a}$, and $x_b \in X^{s_b}$. Moreover, suppose that*

$$d(X^{s_a}, X^{s_b}) = d(x_a, x_b)\,.$$

Then, $[\, x_a, x_b\,]$ can be extended to a geodesic line which is preserved $s_a s_b$. [2]

Proof. This follows from repeated application of the above lemma. $\square$

[1]In CAT(0) spaces, this point is unique, but we will try to stick with our trend of using as few CAT(0) facts as possible

[2]This is the usual translation line associated to a hyperbolic isometry of a CAT(0) space

Corollary 5.1.6. *$\mathcal{T}_i = \mathcal{T}_i^B$ if and only if s_i is central. Moreover, there exist a unbounded component if and only if W is infinite.*

Proof. If there exists a and b with $a \bullet b$, then the relevant translation line must be an infinite Hausdorff distance from both X^{s_a} and X^{s_b} (since it touches both X^{s_a} and X^{s_b}). Thus T_{ab}^y is not bounded. The second part is now obvious. □

The following will be used enough to earn it the title of theorem.

Theorem 5.1.7 (The Wrong Way Theorem)**.** *For any $i \in I$,*

$$\mathcal{T}_i^U = \mathcal{O}\mathrm{T}_i \quad \textit{and} \quad \mathcal{T}_i^B = \mathcal{R}_i.$$

Proof. Since both sides of the second equality are just the compliments of sides of the first equality, the two equalities are equivalent. We will show that the second equality holds. Suppose that $T \in \mathcal{R}_i$. By Corollary 4.4.8, we know that the W-orbit of $x \in T$ meets T if and only if the acting group element is in $W_{\mathsf{c}(i)}$. Assume that T is not i-bounded and consider a geodesic ray in T. Now, let x_t be a point in this ray that lies on a geodesic ray a distance t away from X^{s_i}, and let K be the cocompactness constant for the action. It is easy to see that $d(x_{t+2K}, \mathcal{O}x_t) \geq 2K$; for if $w \notin W_{\mathsf{c}(i)}$, then by what was just said

$$d(w \cdot x_t, x_{t+K}) \geq d(X^{s_i}, x_{t+2K}) = t + 2K.$$

On the other hand, if $w \in W_{\mathsf{c}(i)}$, then

$$d(w \cdot x_t, x_{t+2K}) \geq \left| d(w \cdot x_t, X^{s_i}) - d(x_{t+2K}, X^{s_i}) \right| = 2K.$$

This contradicts that K is a cocompactness constant. □

Remark 5.1.8. This is the only direct use of cocompactness made in the proof of the main theorem.

5.2 Corollaries of the Wrong Way Theorem

Super-turning (Corollary 4.3.4) and the above combine to give a score of useful Corollaries. We first restate Proposition 4.4.3 in a form that uses the theorem.

Corollary 5.2.1. *Let $i \in \mathfrak{a}(W)$. Then the sets*

$$\{\, \mathcal{O}(\mathrm{T}^{y}_{ib}) \,,\, \mathcal{O}(\mathrm{T}^{y}_{ib}) \,\}_{b \in \underline{\mathrm{kl}}(i)}$$

form a complete partition of $X \setminus \left(X^{s_i} \cup \mathcal{T}^{B}_{i} \right)$. □

Corollary 5.2.2. *For any $i \in I$ there is a W-equivariant equivalence:*

$$\mathcal{T}^{U}_{i} \approx \coprod_{b \in \underline{\mathrm{kl}}(i)} W_{\mathsf{c}(i)} / W_{T_b}$$

where T_b is the subset of S which generates $\mathrm{St}\, \mathrm{T}^{y}_{ab}$. □

This corollary allows us to apply the entire theory of minimal coset representatives to $\mathcal{T}_i$. Thus, for every $T \in \mathcal{O}(\mathrm{T}^n_{ij})$ we have a unique $w \in W_{\mathsf{lk}(i)}$ of minimal word length with $T = w \cdot \mathrm{T}^n_{ij}$. In this case we call w a *reduced component path*. Of course this is equivalent to saying that

$$w \in (W_{\mathsf{c}(i)})^{\mathsf{mt}(\mathrm{T}^y_{ij}) \cap \mathsf{c}(i)}$$

Proposition 5.2.3. *Suppose that $i \circ j$. If X^{s_i} meets some j-unbounded component T, then X^{s_i} must meet T^y_{jk} for some $k \in I$.*

Proof. By applying the abelianization map we see that every reduced word representing ws_iw^{-1} must contain s_i. Thus the lemma is a consequence of the theorem and the previous corollary. □

Let $\mathcal{T}_{i,j}$ be the set $\mathcal{T}_j{}^{s_i}$. That is $\mathcal{T}_{i,j}$ is the set of j-components with non-empty intersection with X^{s_i}. Notice that $W_{\mathsf{c}(i)} \cap W_{\mathsf{c}(j)}$ acts on this set. The content of the following proposition is similar to that of theorem.

Proposition 5.2.4. *Suppose that $i \circ j$. Given any $T \in \mathcal{T}_{i,j} \cap \mathcal{T}^U_j$ there is a $k \in \mathsf{kl}(j)$ and a $w \in W_{\mathsf{c}(i)} \cap W_{\mathsf{c}(j)}$ with $w \cdot \mathrm{T}^y_{jk} = T$.*

Proof. We can suppose that X^{s_i} meets some j-unbounded component T. By the theorem, there is a $w \in W_{\mathsf{c}(j)}$ and a $k \in \mathsf{kl}(j)$ with $w \cdot T = \mathrm{T}^y_{jk}$ which we take to be a reduced component path.

Since X^{s_i} meets T, we have that $w \cdot \mathrm{T}^y_{ik} = s_i w \cdot \mathrm{T}^y_{ik}$. Thus, $w(\mathrm{St}\,\mathrm{T}^y_{ik}) = s_i w(\mathrm{St}\,\mathrm{T}^y_{ik})$. Since $w \in (W_{\mathsf{c}(i)})^{\mathsf{mt}(\mathrm{T}^y_{ik}) \cap \mathsf{c}(i)}$, we have that $s_i w = ws$ or $w^{-1} s_i w = s$ for some $* \in \mathrm{St}\,\mathrm{T}^y_{jk}$. By abelianizing we see that $s = s_i$. Thus $w \in W_{\mathsf{c}(i)}$. $\square$

Remark 5.2.5. Notice that this proposition implies having $|\mathcal{T}_i| = \infty$ for any given $i \in I$ is fairly common. For instance this happens whenever $\mathsf{lk}(i)$ contains a non-commuting pair and one element of the pair does not stabilize any given element of $\mathcal{T}_i$, (for instance this is always the case in Remark 4.1.7).

Notation 5.2.6. Let F be any of the sub-group or subset valued function associated to RACP, (W, S, X, ρ), defined so far. For any $a \in I$, we let

$$^{a}F := F(W_{\mathsf{lk}(a)}, \mathsf{lk}(a), X^{s_a}, \rho|_{X^{s_a}}).$$

For as many objects as possible, we would like

$$^{a}F = F \cap X^{s_a} \tag{5.1}$$

for set valued functions and

$$^{a}F = F \cap W_{\mathsf{c}(a)} \tag{5.2}$$

for group valued functions. Indeed, this is often the case. However, for the most important object, $\widetilde{\mathrm{T}^y}$, this is not the case. There are exactly two reasons for this

R1 $W_{\mathsf{lk}(i)}$ may contain central elements even when W does not.

R2 For some $[b] \in \underline{\mathrm{kl}}(i)$ (viewed as a class) it is possible that $a \bullet [b]$.

We will deal with this two problems later. For now, we give a lemma which lists some objects that do satisfy (5.1) or (5.2).

Lemma 5.2.7. *Suppose that a, b and c are distinct elements of I with $c \bullet (a \circ b)$. Then*

(*i*) ${}^{a}X^{w} = X^{s_a} \cap X^{w}$,

(*ii*) *For any $T \in {}^{a}\mathfrak{T}_b$, there exists a unique $T' \in \mathfrak{T}_b$ with $T = T' \cap X^{s_a}$,*

(*iii*) ${}^{a}\mathrm{T}^{y}_{bc} = \mathrm{T}^{y}_{bc} \cap X^{s_a}$.

(*iv*) $\mathrm{C}(s_a s_b) = W_{\mathsf{c}(a)} \cap W_{\mathsf{c}(b)}$,

Proof. Statement (*i*) is a general statement about actions. For if $x \in X^{s_a}$ then $w \cdot x = x$ if and only if $x \in X^{w}$ as well. Also, (*ii*) is just as clear and it has (*iii*) as a special case. The only statement of the four which does not come from a simple general notion (but still trivial!) is (*iv*). Its proof does not really fit in here, but since its proof is comes for free from the geometry, we give it anyway. (Here we can take X to be any turned RACP, e.g., $\Sigma(W, S)$). If $w \in \mathrm{C}(s_i s_j)$, then $w \cdot X^{s_a s_b} = X^{s_a s_b}$. But (by turning) $X^{s_a s_b} = X^{s_a} \cap X^{s_b}$. Thus $w \cdot X^{s_a s_b} = X^{s_a s_b}$ if and only if $w \cdot X^{s_a} = X^{s_a}$ and $w \cdot X^{s_b} = X^{s_b}$. The result now follows from Corollary 3.1.2. $\square$

Technically, we need to show the following before going any further:

Proposition 5.2.8. *$(W_{\mathsf{lk}(a)}, \mathsf{lk}(a), X^{s_a}, \rho|_{X^{s_a}})$ is turned and folded.*

Proof. The quadruple $\mathcal{L}(a) := (W_{\mathsf{lk}(a)}, \mathsf{lk}(a), X^{s_a}, \rho|_{X^{s_a}})$ is a pre-RACP by Proposition 1.3.9 (X^{s_i} is non-empty!).

We now show that $\mathcal{L}(a)$ is turned. Let $i \in \mathsf{lk}(a)$ and $T \in {}^a\mathfrak{T}_i$. By the lemma there exists $T' \in \mathfrak{T}_i$ with $T = T' \cap X^{s_a}$. Let $\mathfrak{f}$ be an i flag in $W_{\mathsf{c}(a)}$. If $T{\cdot}w(\mathfrak{f}) = T$, then $T'{\cdot}w(\mathfrak{f}) = T'$ as well (since $T \subseteq T'$). Since (W, S, X, ρ) is turned, we have that $w(\mathfrak{f}) = 1$.

The proof that $\mathcal{L}(a)$ is folded is analogous and just as straightforward. As such it is left to the reader. ☐

Remark 5.2.9. This also proves that the restricted RACP is still an RACP.

Proposition 5.2.10 (Sneak a Peak)**.** *Suppose that $T \in \mathfrak{T}_b$, $a \circ b$, and $a \bullet \mathsf{in}(\mathcal{O}T)$. Then on $\mathcal{O}T$, the Hausdorff distance between X^{s_a} and X^{s_b} is finite.*

Proof. Of course we can assume that T is b-unbounded. We will however show that in the RACP $(W_{\mathsf{lk}(a)}, X^{s_a})$, $T' := T \cap X^{s_a}$ is b-bounded. By the theorem, this is the same thing as showing that on X^{s_a}, $T \cap X^{s_a}$ is not in the orbit of any `yes` or `no` component. Now suppose that $w{\cdot}T' = {}^a\mathrm{T}^y_{bc}$ [3] for some $c \in \mathsf{c}(a)$ and $w \in W_{\mathsf{c}(a)} \cap W_{\mathsf{c}(b)}$. Then obviously, $w{\cdot}T = \mathrm{T}^y_{bc}$ contradicting the hypothesis. ☐

need to figure out how this is used to choose which version I prefer

[3] from now on, we use a prescript to denote that we are working in the RACP $(W_{\mathsf{lk}(a)}, X^{s_a})$

proof of 5.2.10. In this proof we use a bit of trickery. Instead of dealing with (W, X) we deal with $\left(W_{\mathsf{lk}(a)}, X^{s_a}\right)$. We show that in this RACP the components discussed in the hypothesis of the corollary are b-bounded which implies the desired conclusion. Now, to get our foot in the door, we have proposition 1.3.9 on page 27 which says that the CAT(0) space X^{s_a} is acted on geometrically by $W_{\mathsf{c}(a)}$. It is easy to see that $\left(W_{\mathsf{lk}(a)}, X^{s_a}\right)$ is an RACP. We indicate that we are working in this RACP by placing a "^{a}" before relevant objects. We now look at the components of $X^{s_a} \setminus X^{s_a} \cap X^{s_b}$, which is the set ${}^{a}\mathfrak{T}_b$. Now the component T descends to a component

$$^{a}T := T \cap X^{s_a}.$$

Suppose that ^{a}T is the orbit of a `yes` component or a `no` component in $\mathfrak{I}_b(a)$. Then there exists a $w \in W_{\mathsf{c}(b)}$ and $k \in \mathsf{c}(a)$ with $^{a}T{\cdot}w' =^{a} \mathrm{T}^{\epsilon}_{bk}$. But this implies that $T \cdot w = \mathrm{T}^{\epsilon}_{bk}$. But since our space is folded, $T = \mathrm{T}^{\epsilon}_{bk}$. This contradicts the hypothesis that s_a does not commute with anybody whose fix point set lies in T. By the theorem, we know that this implies that $T' = T \cap X^{s_a}$ is a b-bounded component. Which is just another way of saying that on T, $\mathrm{d}_H(X^{s_a}, X^{s_b})$ is finite. $\square$

Remark 5.2.11. If X has the geodesic extension property, then all components are either `yes` components or `no` components since a space with the geodesic extension property cannot have any bounded components.

5.3 How X^{s_i} sits in $\widetilde{\mathrm{T}^y}$

For the purposes of induction, we will be interested in how the proposed fundamental domain $\widetilde{\mathrm{T}^y}$ behaves when it is intersected with X^{s_i} for $i \in I$. Since $(W_{\mathsf{c}(i)}, X^{s_i}, \mathsf{c}(i), \rho|_{X^{s_i}})$ is also a STUFed RACP, we can talk about components, `yes`'s, `no`'s, etc. We will adopt the notation used in the above, that is placing a prescript "$i|$" (or simply an i) before the relevant term. For example, we call the proposed strict fundamental domain on X^{s_i}, ${}^{i}\widetilde{\mathrm{T}^y}$. We wish to compare that with what we would get if we were to look at $\widetilde{\mathrm{T}^y}$ not in terms of our original space X but in terms of $W_{\mathsf{c}(i)} \curvearrowright X^{s_i}$. We now need to discuss the affect bounded components have on a potential strict fundamental domain.

We deal with products with finite groups first. Let us write $W = W' \times \mathbb{Z}_2^n$ for $n \in \mathbb{N}$. It is a basic fact that W acts geometrically on X if and only W' does. As usual, we write $\widetilde{\mathrm{T}^y}$ for the usual proposed strict fundamental domain for W'. Since $\widetilde{\mathrm{T}^y}$ consists of unions and intersections of `yes`'s and `no`'s and fix point sets, and any element of the $\mathbb{Z}_2^n$ factor preserves these, we get that the $\mathbb{Z}_2^n$ factor preserves $\widetilde{\mathrm{T}^y}$. The lemma below is a version of the main theorem. (Notice that in the proof we do not need that X be compact and this lemma will be applied in this context).

Lemma 5.3.1. *If $(\mathbb{Z}_2^n, X)$ is an* RACP*, then there is a strict fundamental domain for the action.*

Proof. We prove the lemma by induction. At every stage, the inductive assumption allows us to assume that the action is faithful. For the $n = 1$ case, the action is just by $\mathbb{Z}_2$, and we have a pairing of components $\{T, s \cdot T\}$ which determines an equivalence relation where each class has exactly two elements in it (since no component is left invariant). We pick one component from each class and union this with the fix point set. This set is clearly a strict fundamental domain for the action and it is convex since $X^s$is.

Suppose the result holds for $\mathbb{Z}_2^{n-1}$. We consider $\mathbb{Z}_2^n$ as a RACS with the usual generating system. Pick any generator $s \in S$. By the induction hypothesis, $\left(W_{\mathsf{lk}(*)} =\right) \mathbb{Z}_2^{n-1} \curvearrowright X^s$ [4] has a strict fundamental domain. Now every element of $\mathcal{T}_*$ has an orbit under $W_{\mathsf{lk}(*)}$which meets this strict fundamental domain. Now pick one such component from every orbit. For every such T, we consider the stabilizer of $\overline{T}$ (we take the closure to insure the component is a closed CAT(0) space) under the action of $W \curvearrowright \mathcal{T}_*$. This stabilizer is also a finite RACG (finite subgroup of $\mathbb{Z}_2^n$ is also a product of $\mathbb{Z}_2$'s), and it also strongly separates. Moreover, $s \notin \operatorname{St} \overline{T} = \operatorname{St} T$ since s leaves invariant no element of $\mathcal{T}_*$. Thus, again by the induction hypothesis, the set has a strict fundamental domain. We pick a copy of this strict fundamental domain which meets X^s. We do this for all such T. We claim that the resulting space is a strict fundamental domain for the entire group.

1. **It is a fundamental domain** since for every $x \in X$, $x \in T$ for some

[4] We remind the reader that "$*$" is the index of s.

$T \in \mathcal{T}_i$ or $x \in X^s$. In the first case we use $\mathrm{St}\,\overline{T} \curvearrowright \overline{T}$ (viewing $\mathrm{St}\overline{T}$ as the set-wise stabilizer, not the point-wise stabilizer) to move us into the strict fundamental domain for $\overline{T}$, and in the second case, we use the link of s to move x into the strict fundamental domain of $W_{\mathsf{lk}(*)} \curvearrowright X^s$.

2. **It is strict** since if $x \in X^s$, then $w \cdot x \in X^s$ implying that if x and $w \cdot x$ are in the proposed strict fundamental domain, then $w \cdot x = x$. On the other hand if $x \in T \in \mathcal{T}_*$, then $w \cdot x = x$ implies that $w \in \mathrm{St}\,T$ (since w preserves components). And now we use the induction hypothesis like we did if $x \in X^s$.
3. **This set is convex** since it is the union of convex sets glued along a convex set.
4. **This set is closed** since it is the union of a locally finite collection of closed sets □

Definition 5.3.2. Loosely, when doing a procedure such as the one above, we call it *gluing* together strict fundamental domains.

Thus, given the set $\widetilde{\mathrm{T}^y}(W', S')$ and $\{s\} = S \setminus S'$ with s central in W, we can define a new set $\widetilde{\mathrm{T}^y}(W, S) = \widetilde{\mathrm{T}^y}(W' \times \langle s \rangle, S' \cup s)$ by taking a strict fundamental domain of the action of $\langle s \rangle$ on $\widetilde{\mathrm{T}^y}(W', S')$. (Notice that the set $\widetilde{\mathrm{T}^y}(W, S)$ depends on the choice of a strict fundamental domain for the finite groups action on $\widetilde{\mathrm{T}^y}(W', S')$.) The following is an exercise in definitions.

Proposition 5.3.3. $\widetilde{\mathrm{T}^y}(W', S')$ *is a strict fundamental domain for* $W' \curvearrowright X$ *if and only if* $\widetilde{\mathrm{T}^y}(W, S)$ *is a strict fundamental domain for* $W \curvearrowright X$.

Proof. Let $M' := \widetilde{\mathrm{T}^y}(W', S')$ and $M := \widetilde{\mathrm{T}^y}(W, S)$. We first show that M is a strict set. If $y = w{\cdot}x$ for $x, y \in M$ and $w \in W$ then $s \in \urcorner(w)$ (since $M \subseteq M'$). However, $s{\cdot}x \in M'$ and $ws{\cdot}s \cdot x = y$ implies that $s{\cdot}x = y$ (since $ws \in W'$). Since M is a strict fundamental domain for the action of s on M', we know that $x = y$.

To see that M is a fundamental domain, we simply use W' to move any element of X into M' and then we use s to move us into M. Since we already showed that the set is convex and closed in the previous proposition, we have the result. ☐

For any RACS, (W, S), the center of W, $\mathfrak{Z}(W)$, is a finite special subgroup. Let $\mathfrak{z}(W) = I(\mathfrak{Z}(W) \cap S)$. Thus, $W_{\mathfrak{z}(W)} = \mathfrak{Z}(W)$. Moreover, we have a splitting $W = \mathfrak{Z}(W) \times \mathfrak{A}(W)$ where $\mathfrak{a}(W) := S \setminus \mathfrak{z}(W)$ and $\mathfrak{A}(W) := W_{\mathfrak{a}(W)}$.

Lemma 5.3.4. *The action of* $\mathfrak{Z}(W)$ *preserves* $\widetilde{\mathrm{T}^y}(\mathfrak{A}(W), X)$.

Proof. This is a direct consequence of Lemma 3.3.1(*iii*) and the definition of $\widetilde{\mathrm{T}^y}$. ☐

Notation 5.3.5. For any RACP, (W, X), let $\Phi(W, X)$ be a strict fundamental domain for the action of $\mathfrak{Z}(W)$ on $\widetilde{\mathrm{T}^y}(W_{\mathfrak{a}(W)}, X)$.

Thus, the above shows us that $\Phi(W, X)$ is a strict fundamental domain for the action of W if and only $\widetilde{\mathrm{T}^y}$ is a strict fundamental domain for the action of

$W_{\mathfrak{a}(W)}$. The same proof carries through to show the following proposition:

Proposition 5.3.6. *Suppose a group G splits as a direct product $G_1 \times G_2$ and $G \curvearrowright X$. If $G_1 \curvearrowright X$ has a strict fundamental domain M and G_2 preserves M and this action has strict fundamental domain M', then M' is a strict fundamental domain for $G \curvearrowright X$.* $\square$

This set is clearly a strict fundamental domain for the action of W if and only if the original $\widetilde{\mathrm{T}^y}$ was a strict fundamental domain for for the original action. This construction will become more an annoyance when we look at this kind of thing on fix point sets. This is because we cannot just take any strict fundamental domain, but we need to take one which matches up with the original one in a nice way.

A similar thing happens when we look at i-bounded components. Clearly, $W_{\mathsf{c}(i)}$ acts geometrically on the set of i-bounded components, and this set is a CAT(0) space in its own right (this actually classifies such components). Moreover, we have by definition of the centralizer that this is a product action in the sense of the above paragraph. Thus again, we have a choice on how to glue our proposed strict fundamental domain on the i -bounded components to the proposed strict fundamental domain of the group action on some larger space. Again, this choice does not matter when we are looking at the whole space, but a smart choice is needed when we are looking at the action on the various fix point sets.

Let us begin to take a closer look at how ${}^{i}\widetilde{\mathrm{T}}^{y}$ relates to $\widetilde{\mathrm{T}}^{y}$ (we will see that it can be made to be the best possible situation, namely that ${}^{i}\widetilde{\mathrm{T}}^{y} = \widetilde{\mathrm{T}}^{y} \cap X^{s_i}$).

Lemma 5.3.7. *For any $a \in I$ and $b, c \in \mathsf{c}(a)$ with $b \bullet c$*

(i) ${}^{a}X^{s_b} = X^{s_a} \cap X^{s_b}$

(ii) ${}^{a}\mathrm{T}^{y}_{bc} = \mathrm{T}^{y}_{bc} \cap X^{s_a}$

Proof. (i) This is just a general statement about restricted actions.

(ii) Let $T = \mathrm{T}^{y}_{bc} \cap X^{s_a}$. Then since T is the intersection of connected spaces it is connected. Moreover, by (i), T contains X^{s_b}. Thus, we have that $T \subseteq {}^{a}\mathrm{T}^{y}_{bc}$. On the other hand, we know that for any $x \in {}^{a}\mathrm{T}^{y}_{bc}$ that we can find a path contained in X^{s_a} from ${}^{a}X^{s_c}$ to x which avoids ${}^{a}X^{s_b}$. This means that it also avoids X^{s_b}. Hence $x \in T$. $\square$

We finally give the definition of $\Psi = \Psi(W, X)$ who will be the strict fundamental domain for the action.

Definition 5.3.8. We define $\Psi(W, X)$ inductively in a top down, then a bottom up manner (the notation Ψ was chosen for its likeness to the letter Y, as in yes, or `yes`). We do this by gluing to together sets $\Phi_{\mathfrak{f}}(W, S, X) := \widetilde{\mathrm{T}}^{y}(W_{\mathsf{c}(\mathfrak{f})}, S(\mathsf{c}(\mathfrak{f}), \mathcal{X}^{\mathfrak{f}})$ for various $\mathfrak{f} \in \mathcal{S}(W, S)$ to get one big set $\Phi(W, S)$.

Next we go from the bottom up to define Ψ by starting at the maximal flags in $\mathcal{S}(W, S)$ (hence with the minimal centralizers). We assume that the construction of $\Psi_{>\mathfrak{f}}$ has already been made. For the finite special subgroup with index set consisting of the $i \in \mathfrak{f}$ who are central in $W_{\mathsf{c}(\mathfrak{f})}$ but not in $W_{\mathsf{c}(\mathfrak{g})}$ for any $\mathfrak{g} \subset \mathfrak{f}$, we quotient $\Phi_{\mathfrak{f}}(W, S, X)$ by the action of this group. Since this group is finite, its action on $\Phi_{\mathfrak{f}}(W, S, X)$ has a strict fundamental domain by 5.3.1. We glue this strict fundamental domain onto the bit of Ψ that has already been constructed but we insist that we choose the strict fundamental domain of $\Phi_{\mathfrak{f}}(W, S, X)$ to meet $\Phi_{\mathfrak{g}}(W, S, X)$ in the maximum possible way for all $\mathfrak{g} \subset \mathfrak{f}$. This is possible and makes good sense by the construction of the strict fundamental domain in Lemma 5.3.1 as well as Propositions 5.2.4, 5.2.3, and 5.2.10. After doing this for all $\mathfrak{f} \in \mathcal{S}(W, S)$ we arrive at our set $\Psi(W, S, X)$

One should think of $\Psi(W, S, X)$ as $\widetilde{\mathrm{T}^y}(W, S, X)$ but with extra stuff added for bounded components and stuff taken away for properly nested centralizers (*i.e.* $\mathsf{c}(i) \subset \mathsf{c}(j)$). By the construction of $\Psi(W, S, X)$ we the following result.

Proposition 5.3.9. $\Psi(W, S, X) \cap X^{s_i} = \Psi(W_{\mathsf{c}(i)}, S(\mathsf{c}(i)), X^{s_i})$. $\square$

Chapter 6

Finishing The Proof

We are now in the home stretch. As with the last chapter, we assume that (W, X) is STUFed. The first thing that we show is that Ψ is a strict set. We "almost" proved this with Corollary 4.4.4. The only piece of the puzzle that we were missing was what we had to add and subtract to get Ψ.

6.1 Strictness

Theorem 6.1.1 (The Main Result Part 1)**.** $\Psi(W, X)$ *is a strict set.*

Proof. Our proof is by induction on $n = |S|$. The results holds for small n since it holds for all finite groups by Lemma 5.3.1.

We now suppose that x and $x \cdot w$ are in $\Psi(W, X)$. We need to show that $x = x{\cdot}w$. If both either are in T^y, then we know that $w = 1$ by Proposition 4.4.4. If both are in $\mathfrak{X}^{s_i}$ for some $i \in I$, then the induction hypothesis and 5.3.9 tell us again that both points in x's orbit are the same.

Lastly we must consider the case that $x \in \mathfrak{X}^{s_i}$ and $x{\cdot}w \in \mathfrak{X}^{s_j}$ for possibly distinct $i, j \in I$ and show that it reduces to the (second) case above. By Corollary 4.4.9 we know that $w = w_1 w_2$ with $w_1 \in W_{\mathsf{c}(i)}$ and $w_2 \in W_{\mathsf{c}(j)}$.

We claim that $w_2 \in W_{\mathsf{c}(i)}$ as well. For if we factor w_2 as $w_{21}w_{22}w_{23}$ with $w_{21} \in W_{\mathsf{c}(i)}$, $w_{22} \in {}^{\mathsf{c}(i)}W^{\mathsf{c}(i)}$, and $w_{23} \in W_{\mathsf{c}(i)}$ we can assume that w_{21} is trivial (by moving it over to join up with w_1). Thus we must show that w_{22} is trivial. By definition, for any $a \in \urcorner(w_{22})$ we have that $a \notin W_{\mathsf{c}(i)}$ and such an a exists if we assume that w_{22} is not trivial. As such since $x{\cdot}w_1 \in \mathfrak{X}^{s_i}$, we must also have that $x{\cdot}w_1$ is in T^y_{ai}. The usual trickery applies to show that $x{\cdot}w$ lies in (the orbit of) a `no` component of all elements of $\urcorner(w^{-1})$. But this contradicts our assumption that $x{\cdot}w \in \Psi(W, X)$. Thus we have that w_{22} as desired.

Combining what we have just shown, we must have that $w \in W_{\mathsf{c}(i)}$. Thus x and $x{\cdot}w$ live in $\mathfrak{X}^{s_i}$ and we are reduced to the case of the first paragraph as promised. $\square$

6.2 The Organization

Recall in the Davis complex, if you are not in the strict fundamental domain, you are in the `no` component of the last acting generator. In the more general case, this is not true when bounded components or non-trivial wedge pairs exist. However, the next best thing does exist. That is if we are not in our first approximation of a strict fundamental domain, $\widetilde{\mathrm{T}^y}$, then we are in a wedge of `no` components or a bounded component. This description is more suitable for metric considerations.

Proposition 6.2.1. *If*

$$x \notin \widetilde{\mathrm{T}^y} \bigcup_{i \in I} \mathcal{X}^{s_i},$$

then there exists a k-wedge tuple over the flag $\mathfrak{f}$ with $x \in \mathrm{T}^n_{\mathfrak{f}k}$.

Proof. By Theorem 5.1.7 above, it suffices to show that every

$$x \in \mathcal{O}\mathrm{T}^n_{ab} \cup \mathcal{O}\mathrm{T}^y_{ab}$$

is contained in such a wedge. The proof is by induction on the number $\ell(w) = m$ with $w = s_1 \cdots s_m$. By assumption, we have that $x \notin \mathrm{T}^y_{ab}$ for some $a, b \in I$. We deal with the case where x lives in an orbit of T^y_{ab}, the case that x lives in the orbit of T^n_{ab} is similar and even slightly easier. Thus, we are saying that $x \in \mathrm{T}^y_{ab} \cdot w$. Now if $\ell(w) = 1$, then $\mathrm{T}^y_{ab} \neq \mathrm{T}^y_{ab} \cdot s_1$ (since we are assuming that x

is not in T^{y}_{ab}). By Lemma 3.3.1, we know that $\mathrm{T}^{y}_{ab}\cdot s_1 \subseteq \mathrm{T}^{n}_{1b}$. Hence, $x \in \mathrm{T}^{n}_{1b}$, and the results holds for $m = 1$.

Now if the result holds for $m-1$, then

$$x\cdot s_m \in \mathrm{T}^{y}_{ab}\cdot(s_1\cdots s_{m-1}) \subseteq \bigwedge_{i\in\mathfrak{f}} \mathrm{T}^{n}_{ib}.$$

Where $\mathfrak{f} = \{i_1, \ldots, i_f\}$ is a flag $1 \leq i_1 \cdots \leq i_f \leq (m-1)$ and $1 \leq f \leq (m-1)$.

We have two cases to consider. Either s_m is flag to $\mathfrak{f}$ or it is not. If it is flag to $\mathfrak{f}$, then we will have four subcases to consider. Namely, the combinations of how the flag and s_m stabilize each others relevant components.

First, if m is not flag to $\mathfrak{f}$, then by definition there is an $i \in \mathfrak{f}$ so that s_m and s_i do not commute. Lemma 4.3.3 guarantees that

$$\left(\bigwedge_{i\in\mathfrak{f}} \mathrm{T}^{n}_{ib}\right)\cdot s_m \subseteq \mathrm{T}^{n}_{mi}.$$

On the other hand, if m is flag to $\mathfrak{f}$, then we only need consider the case where

$$\left(\bigwedge_{i\in\mathfrak{f}} \mathrm{T}^{n}_{ib}\right)\cdot s_m \neq \bigwedge_{i\in\mathfrak{f}} \mathrm{T}^{n}_{ib}.$$

By Proposition 3.4.2, we have that

$$\mathrm{T}^{y}_{ib}\cdot s_m \neq \mathrm{T}^{y}_{ib} \text{ for all } i \in \mathfrak{f} \tag{6.1}$$

(since $\mathrm{T}^{y}_{i_1 b} = \mathrm{T}^{y}_{i_2 b}$ for all $i_1, i_2 \in \mathfrak{f}$).

Thus, we only need to consider the two remaining subcases where either the flag stabilizes T^{y}_{mb} or it does not. Now, if for any $i \in \mathfrak{f}$ we have that $\mathrm{T}^{y}_{mb}\cdot s_i \neq \mathrm{T}^{y}_{mb}$, then we must have that T^{y}_{mb} and T^{y}_{jb} are a wedge pair. Thus, by Corollary 3.4.4 we know that

$$\left(\bigwedge_{i\in\mathfrak{f}} \mathrm{T}^{n}_{ib}\right) \wedge \mathrm{T}^{n}_{mb}$$

makes sense and

$$\left(\bigwedge_{i\in\mathfrak{f}} \mathrm{T}^{n}_{ib}\right)\cdot s_m = \left(\bigwedge_{i\in\mathfrak{f}} \mathrm{T}^{n}_{ib}\right) \wedge \mathrm{T}^{n}_{mb}.$$

The last case to consider is if the flag stabilizes T^{y}_{mb} but s_m does not stabilize the wedge. Thus, in addition to (6.1), we are assuming

$$\mathrm{T}^{y}_{mb}\cdot s_i = \mathrm{T}^{y}_{mb} \text{ for all } i \in \mathfrak{f}. \tag{6.2}$$

$$\begin{aligned} \text{So,} \quad \mathrm{T}^n_{\mathfrak{f}b}\cdot s_m &= \mathrm{T}^y_{\mathfrak{f}b}\cdot(w(\mathfrak{f})s_m) && \text{(By definition of a wedge)} \\ &= \mathrm{T}^y_{\mathfrak{f}b}\cdot(s_m w(\mathfrak{f})) && \text{(m is flag to $\mathfrak{f}$)} \\ &\subseteq \mathrm{T}^n_{mb}\cdot w(\mathfrak{f}) && \text{(By (6.1))} \\ &= \mathrm{T}^n_{mb}. && \text{(By (6.2))} \end{aligned}$$

Thus, in all cases the result holds. □

6.3 Fundamental Domain

First a final simple lemma:

Lemma 6.3.1. *For every maximal flag $\mathfrak{f} = \{i_1, \dots, i_m\}$, $X^{\mathfrak{f}}$ is a non-empty compact subset of $\widetilde{\mathrm{T}^y}$.*

Proof. Since (W, X) is properly nested and $w(\mathfrak{f})$ has finite order, $X^{\mathfrak{f}} = X^{w(\mathfrak{f})}$ and $X^{\mathfrak{f}} \neq \emptyset$. Now, suppose that $j \in I$. Since $\mathfrak{f}$ is maximal, if $j \circ \mathfrak{f}$, then $j \in \mathfrak{f}$ and $X^{\mathfrak{f}} \subseteq X^{s_j}$. On the other hand if not, then $j \bullet i$ for some $i \in \mathfrak{f}$ and $X^{\mathfrak{f}} \subseteq X^{s_i} \subseteq \mathrm{T}^y_{ji}$. Thus,

$$X^{\mathfrak{f}} \subseteq \widetilde{\mathrm{T}^y_j} \quad \text{for all } j \in I,$$

and therefore

$$X^{\mathfrak{f}} \subseteq \widetilde{\mathrm{T}^y}.$$

Lastly, we know by Proposition 1.3.9 that the centralizer of $w(\mathfrak{f})$ acts geometrically on $X^{w(\mathfrak{f})}$. However, the centralizer is a finite group — actually, $\mathrm{C}(w(\mathfrak{f})) = W_{\mathfrak{f}}$ — which implies that $X^{w(\mathfrak{f})} = X^{\mathfrak{f}}$ is compact (only a compact space can admit a cocompact group action by a finite group). $\square$

And finally:

Theorem 6.3.2 (The Second Part)**.** *The set Ψ is a fundamental domain (hence a strict fundamental domain when coupled with Theorem 6.1.1).*

Proof. Let $x \in X$. If $x \in \widetilde{\mathrm{T}^y}$, then we can use central elements to move us into Ψ and we are done. Thus, we assume that $x \notin \widetilde{\mathrm{T}^y}$. Our proof is by induction on the total number of generators of W. If this number is 1, then we are in the finite case which was covered in Lemma 5.3.1. Thus, we assume that the theorem holds for all RACP's with generating sets with less than or equal to a given number of elements. If $x \in \mathfrak{X}^{s_i}$ for some $i \in I$ then we use the induction hypothesis and Proposition 5.3.9 to use the link of i to move x into $\Psi(W_{\mathsf{c}(i)}, S(\mathsf{c}(i)), \mathfrak{X}^{s_i})$. This set is contained in $\Psi(W, S, X)$ by Proposition 5.3.9. Thus, we assume that $x \notin \mathfrak{X}^{s_i}$ for any $i \in I$.

Now, by Proposition 6.2.1, we know that

$$x \in \mathrm{T}^n_{\mathfrak{f}k} := T_o$$

for some flag $\mathfrak{f}$. Now let $\mathfrak{g}$ be a maximal flag. If $\mathfrak{f} \subseteq \mathfrak{g}$, then we have that

$$d(x, X^{\mathfrak{g}}) = d(w(\mathfrak{f})\cdot x, X^{\mathfrak{g}})\,.$$

On the other hand, if $\mathfrak{f} \not\subseteq \mathfrak{g}$, then there is an $i \in \mathfrak{f}$ so that $i \notin \mathfrak{g}$. Thus, by maximality of $\mathfrak{g}$, there is a $j \in \mathfrak{g}$ with $i \bullet j$. Hence,

$$X^{\mathfrak{g}} \subseteq X^{s_j} \subset \mathrm{T}^y_{ij}.$$

Moreover, $\mathrm{T}^y_{ij} \neq T_o$ since our space is folded. Thus we must have that

$$d(x, X^{\mathfrak{g}}) < d(w(\mathfrak{f})\cdot x, X^{\mathfrak{g}})$$

if j and k determine the same i-component (by convexity of the sets $\widetilde{\mathrm{T}^y}$) and

$$d(x, X^{\mathfrak{g}}) = d(w(\mathfrak{f})\cdot x, X^{\mathfrak{g}})$$

if they determine different i-components (by a slight variation of Corollary 1.3.21).

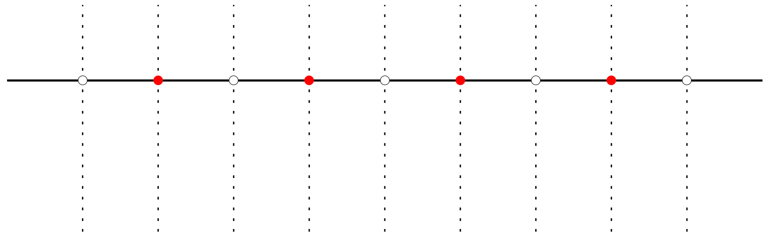
Now, if $w(\mathfrak{f})\cdot x$ is still not in an extended fixed point set or $\widetilde{\mathrm{T}^y}$, then we could do the same procedure. By 6.3.1 maximal flags are compact and lie in $\widetilde{\mathrm{T}^y}$. Hence, we must eventually enter $\widetilde{\mathrm{T}^y}$ or an extended fixed point set of some generator. If we enter an extended i-fixed point set, then as above we use the link to move us into $\Psi(W_{\mathsf{c}(i)}, \mathsf{c}(i), X^{s_i})$ by the induction hypothesis. Finally, since $\Psi(W, X)$ is a fundamental domain for the action of $\mathfrak{Z}(W)$ on $\widetilde{\mathrm{T}^y}$, as we already pointed out in the first paragraph of this proof, we can use the center of W to move us into $\Psi(W, X)$ $\square$

Appendix A

Some Examples

In this appendix, we present several simple examples that highlight many of the important ideas of this document. For the most part, this section can be read independently of the rest of the paper (except for the notation). Most actions are fairly simple actions on the tree of valence four, however these actions display nearly the entire range of pathologies that are dealt with extensively throughout this thesis. Moreover these sorts of pathologies extend to a wide range of spaces, the only thing that does not extend is the author's ability to draw these spaces.

Example A.1 (Usual Infinite Dihedral Action). This is the most basic example of an infinite RACG. It acts on the real line with one generator acting as $x \mapsto -x$ and the other as $x \mapsto 1 - x$.



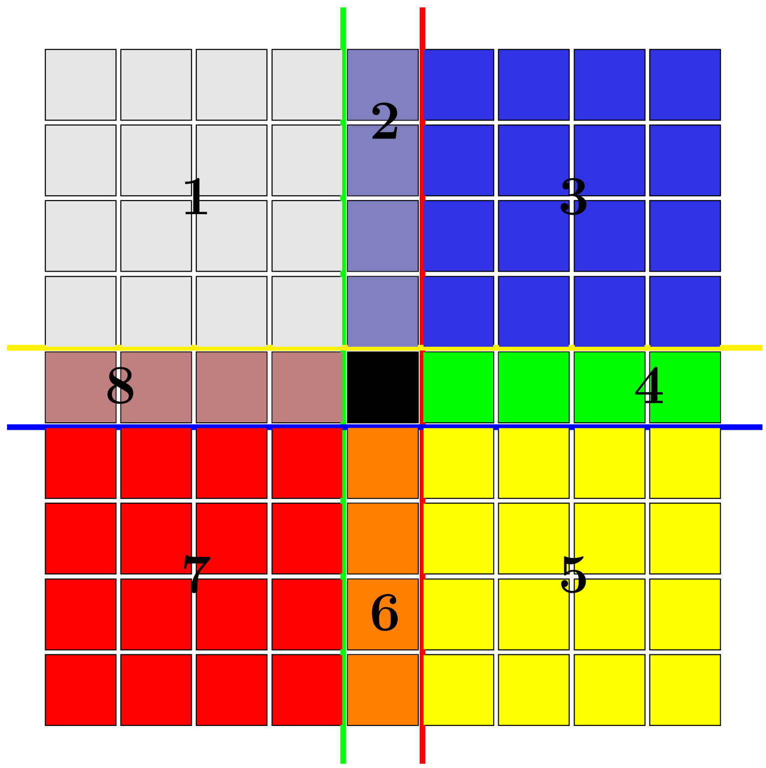
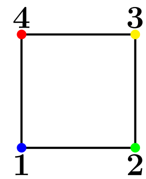


Figure A.1: The corresponding image for Example A.2. The indicated sets are (up to a set of measure zero) $1 = \mathrm{T}_2^n \cap \mathrm{T}_3^n$, $2 = \mathrm{T}_3^n \setminus \mathrm{T}_2^n \cup \mathrm{T}_4^n$, etc.The fixed point sets match up with the generators in the defining graph.

Example A.2 (Usual Planar Action). The second most basic example of an infinite RACP is determined by the action of the RACG defined by the graph



This group is just the direct product of two copies of the infinite dihedral group, one copy generated by s_1, s_3 and the other generated by s_2, s_4. Even in this simple case we can see how the action divides up the space into the indicated regions, each of which has a strong algebraic significance, namely they are the regions where if (an interior point) of the strict fundamental domain meets them by the action of an element $w \in W$, then $\urcorner(w)$ is equal to precisely the corresponding `no` components. (This of course is always true for the Davis complex or all of the other usual constructions one sees in Coxeter groups, e.g., the Tits cone, the Coxeter complex, etc.)

Example A.3. Here we are looking at the action of the free product of 3 copies of the cyclic group of order 2 on the regular tree of valence 4. This is the first example presented in this special example section of an RACP that is not the same as the Davis complex, and it will no doubt be one of the first examples that the reader will compare the results arrived at here with.

The important thing to notice is that (taking the generators as in Figure A.2) $\mathrm{T}^y_{21} \neq \mathrm{T}^y_{23}$. Moreover, $\mathcal{T}_2$ has four elements and not two.

Example A.4. This is the example referenced in 3.2. Here the action is as indi-

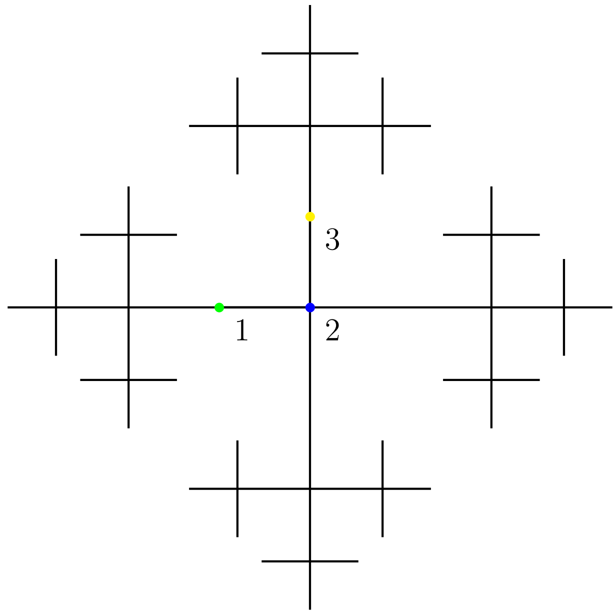


Figure A.2: The space for example Example A.3

cated in Figure A.3. It is also the basic example of the phenomena described in Remark 4.1.7 on page 67 with the underlying group being the free product of three copies of the cyclic group of order 2 and the partial conjugation being any of the form $\mathrm{Pc}_{s,\{t\}}$ with distinct s and t in S. These come together to form the extension group

The fixed point sets are color coordinated with the defining graph. The action of the individual elements are the obvious reflections. The strict fundamental domain for the action is indicated in black. Notice that the fixed point sets of both s_2 and s_3 separate the space into an infinite number of components. Moreover every point in the space lives in the fixed point set of some reflection.

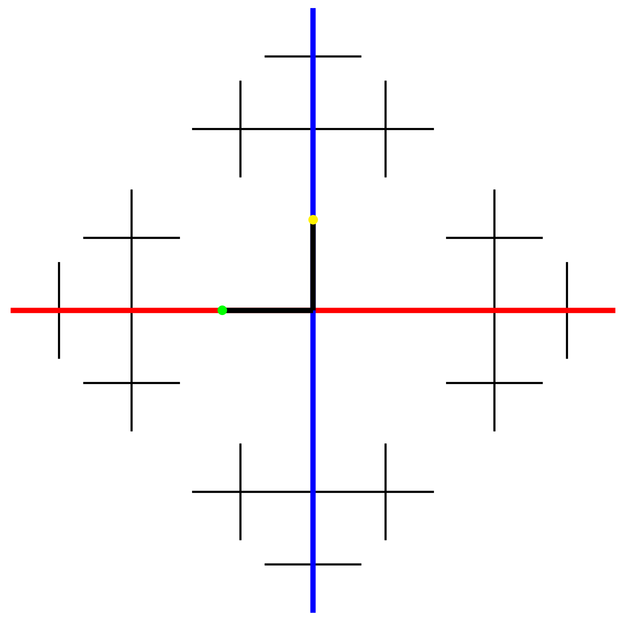


Figure A.3: The graph corresponding to the action described in Example A.4

Example A.5. This illustrates the folding procedure of Section 4.3 and outlined in Section 4.1.3. The tree in Figure A.4 shows the action of four generators on the regular tree of valence four. The action is really the same as the action of the Davis complex in disguise as the reader can convince themselves by applying the folding construction described.

Example A.6. This example shows how a space can have the geodesic extension property, but the fix point sets do not. The action is again by the free product of three copies of the cyclic group of order 2, but this time it is acting on

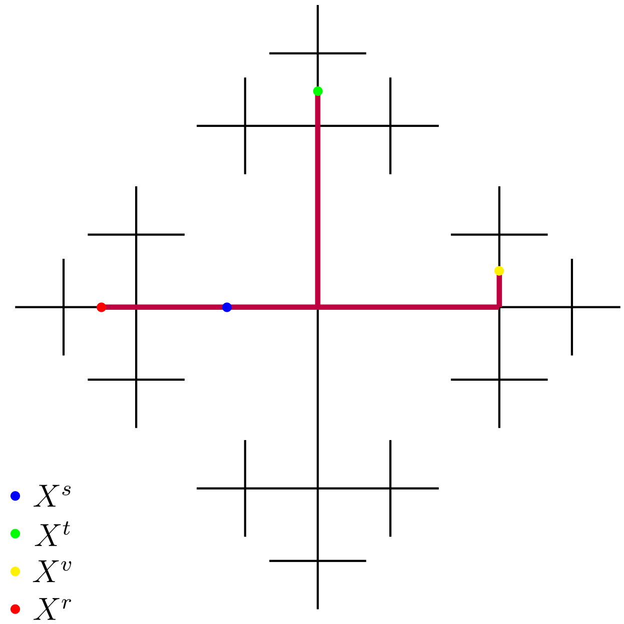


Figure A.4: The figure corresponding to Example A.5

$\mathbb{T}_3$. However, this is not the usual action. In this action, we require that the generators s_1 and s_2 act as in the Davis complex action (flipping around the midpoints of the edges $[v_{11}, v_1]$ and about the edge $[v_{12}, v_1]$ resp.), but the third generator acts by fixing the entire edge $[v_1, v_2]$ as indicated in Figure (NEED TO ADD AN IMAGE HERE)

Example A.7. Let the infinite dihedral group act on the Euclidean unit disk by reflections where the angle between the reflecting lines is an irrational multiple of π. Then the action is not proper and does not have a strict fundamental domain. This is in spite of the fact that the action satisfies all of the other requirements to be an RACP. It is easy to see that the space does not have a strict fundamental domain since any strict fundamental domain for the action would have empty interior, which is impossible.

Appendix B

Index of Notation

Here are some commonly used (*i.e.*, all but the ones that I forgot to write down) symbols and notations. Here, X is a CAT(0) space, $A \subseteq X$, $s \in S$, and $i, j \in I$. Moreover, we may add a subscript to any of the mentioned without changing their description. For example, $s_i \in S$. Moreover, we put a prime on any sub-objects. For example, $S' \subseteq S$. Also, $\mathfrak{f}$ is a flag.

$\mathrm{C}_G(g)$	The centralizer of an element g in the group G.
$\mathfrak{Z}(G)$	The center of a group G.
$\Gamma(W,S)$	The Coxeter diagram of the Coxeter system (W,S).
$\mathcal{T}_i$	The set of connected components of $X \setminus X^{s_i}$.
$\mathsf{in}(A)$	$\{\, i \in I \mid X^{s_i} \subseteq A \,\}$.
$\mathsf{mt}(A)$	$\{\, i \in I \mid X^{s_i} \cap A \neq \emptyset \,\}$.
T_{ij}^y	The unique element of $\mathcal{T}_i$ which contains X^{s_j}.
T_{ij}^n	$s_i \cdot \mathrm{T}_{ij}^y$.
T_i^n	$s_i \cdot \mathrm{T}_i^y$.

$w(\mathfrak{f})$	$\prod_{i\in\mathfrak{f}} s_i$ (i.e, the maximal element of $W_{\mathfrak{f}}$)
T_i^y	$\bigcup_{k\in\mathsf{kl}(i)} \mathrm{T}_{ij}^y$.
T^y	$\bigcap_{i\in I} \mathrm{T}_i^y$.
$\widetilde{\mathrm{T}^y}$	$\bigcap_{i\in I} \mathrm{T}_i^y \cup X^{s_i}$.
X^{s_i}	The fix point set of s_i under an action on X.
S	As in (W,S).
W	As in (W,S).
I	The index set of S.
$w_1 \circ w_2$	w_1 and w_2 commute.
$w_1 \bullet w_2$	w_1 and w_2 do not commute.
$i \circ j$	s_i and s_j commute.
$w \circ I'$	w commutes with s_i for every $i \in I'$.
$w \circ S'$	w commutes with every $s \in S'$.
$i \bullet j$	s_i and s_j do not commute.
$A \subset B$	A is a subset of B and $A \neq B$.
$A \subseteq B$	A is a subset of B.
$G^\rho \cdot A$	The orbit of A under an action ρ of G on X.
$\mathsf{lk}(i)$	The graph theoretical link of v_i in $\Gamma(W,S)$.
i	The graph theoretical star of v_i in $\Gamma(W,S)$.
$\mathcal{O}_i A$	The orbit of A under the action of $W_{\mathsf{lk}(i)}$.
ϵ	An element of $\{y,n\}$.
$\mathrm{T}_{\mathfrak{f}k}^y$	T_{ik}^y where i is any element of $\mathfrak{f}$ (defined only when $\mathfrak{f}$ is a wedge k-tuple).
$\mathrm{T}_{\mathfrak{f}k}^n$	$w(\mathfrak{f})\cdot \mathrm{T}_{\mathfrak{f}k}^y$.
$\mathcal{L}(w)$	The set of letters appearing in any reduced word word representing w.

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