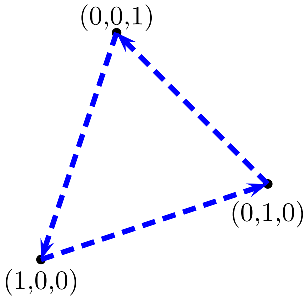
Answer the questions in the exam booklet. Answers provided on this sheet will be ignored.

1. (15 points) For this problem let
$$\overrightarrow{F}(x,y,z) = z\widehat{k}.$$
   (a) (5 points) By calculating the curl or explicitly finding a function, verify that $\overrightarrow{F}$ is the gradient of some function.

   (b) (5 points) Directly compute the line integral
$$\int_{\mathcal{C}} \overrightarrow{F} \bullet d\overrightarrow{s}$$
   where $\mathcal{C}$ is the boundary curve of a triangle, $\mathcal{W}$, with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ transversed in that order.



   (c) (5 points) Explain how we could have known the answer to the previous part without doing a calculation.

2. (15 points) Let $\mathcal{W}$ be the same triangle from the previous problem but for this problem we instead let
$$\overrightarrow{F}(x,y,z) = x\widehat{k}.$$
   Verify Stokes theorem for this vector field and this surface.

Sol'n

1. Surface Integral. The curl of this vector field is $-\widehat{\jmath}$. The surface is parametrized as
$$\begin{aligned}\Phi(u,v) =& (u,v,1-u-v)\\ & 0 \le v \le 1-u\\ & 0 \le u \le 1.\end{aligned}$$
   Here $\Phi_u = \widehat{\imath} - \widehat{k}$ and $\Phi_v = \widehat{\jmath} - \widehat{k}$ which has a cross product
$$n(u,v) = \Phi_u \times \Phi_v = (\widehat{\imath} - \widehat{k}) \times (\widehat{\jmath} - \widehat{k}) = \widehat{\imath} + \widehat{\jmath} + \widehat{k}.$$

(Notice that the orientations match up so we can apply Stokes theorem.)

$$\begin{aligned}\iint_T (\nabla \times x\widehat{k}) \bullet \, d\overrightarrow{S} &= \int_0^1 \int_0^{1-u} (-\widehat{j}) \bullet (\widehat{i} + \widehat{j} + \widehat{k}) \, dvdu \\ &= -\int_0^1 \int_0^{1-u} \, dvdu \\ &= -\int_0^1 (1-u)du \\ &= \int_0^1 (u-1)du \\ &= -\frac{1}{2}.\end{aligned}$$

2. Line Integral. The boundary of the triangle has three portions:

$$c_1(t) = (1-t, t, 0) \qquad c_2(t) = (0, 1-t, t) \qquad c_3(t) = (t, 0, 1-t)$$

(each with parameter running between 0 and 1). The 3 corresponding velocity vectors are

$$\overrightarrow{c}'_1(t) = -\widehat{i} + \widehat{j} \qquad \overrightarrow{c}'_2(t) = \widehat{j} + \widehat{k} \qquad \overrightarrow{c}'_3(t) = \widehat{i} - \widehat{k}.$$

Since $\overrightarrow{F}(c_2(t)) = \overrightarrow{0}$ we needn't consider the second portion of the triangle. Similiarly $\overrightarrow{c}'_1(t)$ is everywhere perpendicular with $\overrightarrow{F}(c_1(t))$ so we only need to consider the integral over the third portion. That is

$$\int_{\mathcal{C}} \overrightarrow{F} \bullet d\overrightarrow{s} = \int_{\mathcal{C}_3} \overrightarrow{F} \bullet d\overrightarrow{s}.$$

There we have

$$\overrightarrow{F}(c_3(t)) \bullet \overrightarrow{c}'_3(t) = (1-t)(-1) = t-1$$

and the line integral is therefore

$$\int_0^1 (t-1) \, dt = -\frac{1}{2}$$

which is the same answer as we got by doing the surface integral.

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3. (30 points) (a) (5 points) Parametrize a cone of radius R and height H which is situated as on the board.
   (b) (5 points) Compute a normal vector to this cone which points outwards.
   (c) (5 points) Show that the surface area of this cone (not including the top of the cone) is

   $$\pi R\sqrt{R^2 + H^2}.$$

   (d) (5 points) Compute the suface integral

   $$\iint_{\text{Cone}} \overrightarrow{F} \bullet \mathrm{d}\overrightarrow{S}$$

   over this cone where

   $$\overrightarrow{F}(x, y, z) = x\widehat{i}.$$

   (e) (5 points) Compute the volume enclosed by this cone by using the previous part and the divergence theorem. Please be clear how exactly you are using the divergence theorem. For instance why can you ignore the top of the cone?

(f) (5 points) Compute the volume instead by doing a triple integral.

4. (20 points) Find the volume contained inside of an ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

by doing a triple integral and changing coordinates via

$$\Phi(x, y, z) = (ax, by, cz).$$

Sol'n

Notice that the mapping Φ takes the ball of radius 1 to the inside of ellispoid, denoted $\mathcal{E}$, in question. That is $\Phi(\mathbb{S}^2) = \mathcal{E}$. It has Jacobian determinant equal to abc. Therefore, by the change of variables formula we have

$$\mathrm{Vol}(\mathcal{E}) = \iiint_{\mathcal{E}} dV = \iiint_{\Phi(\mathbb{S}^2)} dV = \iiint_{\mathbb{S}^2} \det Jac(\Phi) dV = abc \iiint_{\mathbb{S}^2} dV = abc \cdot \mathrm{Vol}(\mathbb{S}^2) = \frac{4}{3}\pi abc.$$

5. (20 points) Let $\mathcal{W}$ be the portion of the unit sphere in the first octant. In other words, we are considering the region

$$\begin{aligned} &x^2 + y^2 + z^2 \le 1 \\ &x \ge 0 \\ &y \ge 0 \\ &z \ge 0. \end{aligned}$$

Compute the integral

$$\iiint_{\mathcal{W}} xyz \; dxdydz$$

in your favorite two coordinate systems (two of cartesian, cylindrical, or spherical).

Sol'n

1. Spherical.
   In spherical coordinates the region, $\mathcal{W}$ is described as

$$\begin{aligned} &0 \le \rho \le 1 \\ &0 \le \theta \le \pi/2 \\ &0 \le \phi \le \pi/2, \end{aligned}$$

the integrand

$$xyz = (\rho\cos\theta\sin\phi) \cdot (\rho\sin\theta\sin\phi) \cdot (\rho\cos\phi) = \rho^3\cos\theta\sin\theta\sin^2\phi\cos\phi,$$

and as always (in spherical coordinates)

$$dxdydz = \rho^2\sin\phi \, d\rho \, d\theta \, d\phi.$$

Hence

$$\begin{aligned}\iiint_{\mathcal{W}} xyz\ dxdydz &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \left(\rho^3\cos\theta\sin\theta\sin^2\phi\cos\phi\right)\cdot\left(\rho^2\sin\phi\, d\rho\, d\theta\, d\phi\right)\\ &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho^5)(\cos\theta\sin\theta)(\sin^3\phi\cos\phi)\, d\rho\, d\theta\, d\phi.\\ &= \left(\int_0^1 \rho^5\, d\rho\right)\left(\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\right)\left(\int_0^{\pi/2}\sin^3\phi\cos\phi\, d\phi\right)\end{aligned}$$

To reduce clutter let's denote the three terms in the product by I_1, I_2 and I_3. Therefore we need to find these and multiply them to get our final answer.

$$\begin{aligned}I_1 &= \int_0^1 \rho^5\, d\rho = \frac{\rho^6}{6}\Big|_0^1 = \frac{1}{6}.\\ I_2 &= \int_0^{\pi/2}\cos\theta\sin\theta\, d\theta = \int_0^1 u\, du = \frac{1}{2}\\ I_3 &= \int_0^{\pi/2}\sin^3\phi\cos\phi\, d\phi = \int_0^1 u^3\, du = \frac{1}{4}.\end{aligned}$$

Therefore the answer is

$$I_1\cdot I_2\cdot I_3 = \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{48}.$$

2. Cylindrical.

In spherical coordinates the region, $\mathcal{W}$ is described as

$$\begin{aligned}0 &\le z \le \sqrt{1-r^2}\\ 0 &\le \theta \le \pi/2\\ 0 &\le r \le 1.\end{aligned}$$

the integrand

$$xyz = (r\cos\theta)\cdot(r\sin\theta)\cdot(z) = \mathcal{R}^2\cos\theta\sin\theta z,$$

and as always (in cylindrical coordinates)

$$dxdydz = r\, dr\, d\theta\, dz.$$

Hence

$$\begin{aligned}
\iiint_{\mathcal{W}} xyz\ dxdydz &= \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} \left(r^2\cos\theta\sin\theta z\right)\cdot(r\,dz\,dr\,d\theta)\\
&= \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} zr^3\cos\theta\sin\theta\,dz\,dr\,d\theta\\
&= \int_0^{\pi/2}\int_0^1 \frac{z^2}{2}\Big|_0^{\sqrt{1-r^2}} r^3\cos\theta\sin\theta\,dr\,d\theta\\
&= \frac{1}{2}\int_0^{\pi/2}\int_0^1 (1-r^2)\,r^3\cos\theta\sin\theta\,dr\,d\theta\\
&= \frac{1}{2}\int_0^{\pi/2}\int_0^1 (r^3-r^5)\cos\theta\sin\theta\,dr\,d\theta\\
&= \frac{1}{2}\int_0^{\pi/2}\left(\frac{r^4}{4}-\frac{r^6}{6}\right)\Big|_0^1 \sin\theta\cos\theta\,d\theta\\
&= \frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{4}-\frac{1}{6}\right)\sin\theta\cos\theta\,d\theta\\
&= \frac{1}{2}\int_0^{\pi/2}\frac{1}{12}\sin\theta\cos\theta\,d\theta\\
&= \frac{1}{24}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\\
&= \frac{1}{48}.
\end{aligned}$$

(Which is the same answer as we got by doing the problem in spherical coordinates).

3. Cartesian. In cartesian coordinates the region, $\mathcal{W}$ is described as

$$\begin{aligned}
&0\le z\le\sqrt{1-x^2-y^2}\\
&0\le y\le\sqrt{1-x^2-y^2}\\
&0\le x\le 1.
\end{aligned}$$

Hence

$$\begin{aligned}
\iiint_{\mathcal{W}} xyz\ dxdydz &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dzdydx \\
&= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\ dydx \\
&= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy - x^3y - xy^3\ dydx \\
&= \frac{1}{2} \int_0^1 \left(\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4} \right) \Big|_0^{\sqrt{1-x^2}}\ dx \\
&= \frac{1}{4} \int_0^1 \left(x(1-x^2) - x^3(1-x^2) - \frac{x(1-x^2)^2}{2} \right)\ dx \\
&= \frac{1}{4} \int_0^1 \left(x - x^3 - x^3 + x^5 - \frac{x(1-2x^2+x^4)}{2} \right)\ dx \\
&= \frac{1}{4} \int_0^1 \left(x - x^3 - x^3 + x^5 - \frac{(x-2x^3+x^5)}{2} \right)\ dx \\
&= \frac{1}{4} \int_0^1 \left(\frac{x}{2} - x^3 + \frac{x^5}{2} \right)\ dx \\
&= \frac{1}{4} \cdot \frac{1}{12} \\
&= \frac{1}{48}.
\end{aligned}$$

(Hence we, for the third time, get an answer of $\frac{1}{48}$).

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6. (50 points) **Bonus!!**

Let ω be the 1-form

$$\omega = yz\,dx + xz\,dy + xy\,dz$$

compute $d\omega$.

Sol'n This was just a check to see if anybody started working on the bonus already. Taking d of a 1-form is exactly the same operation as doing the curl of the corresponding vector field.

$$\begin{aligned}
d\omega &= d(yz\,dx + xz\,dy + xy\,dz) = d(yz\,dx) + d(xz\,dy) + d(xy\,dz) \\
&= (z\,dy + y\,dz) \wedge\ dx + (z\,dx + x\,dz) \wedge\ dy + (y\,dx + x\,dy) \wedge\ dz \\
&= z\,dydx + y\,dzdx + z\,dxdy + x\,dzdy + y\,dxdz + x\,dydz \\
&= -z\,dxdy + y\,dzdx + zdxdy - x\,dydz - y\,dzdz + x\,dydz \\
&= 0\,dydz + 0\,dzdx + 0\,dxdy. \\
&= 0.
\end{aligned}$$
