Answer the questions in the exam booklet. Answers provided on this sheet will be ignored. Clearly label your work and any problems skipped. If I cannot read an answer or your work, I will assume it's wrong!

1	2	3	4	5	6	7	8	9	10	11	Total
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1. (10 points) Which pairs of the following vectors are orthogonal (perpendicular) to each other?

(i) $(1,2,3)$ (iii) $\widehat{i}-2\widehat{j}+\widehat{k}$ (v) $\widehat{i}-2\widehat{j}-5\widehat{k}$

(ii) $2\widehat{i}+\widehat{j}$ (iv) $(4,1,-2)$ (vi) $(1,1,2)$

Sol'n

The table below displays the various dot products (since the dot product is symmetric and a vector dot itself is positive, there is no need to evaluate the dot product in places indicated with an asterix, $*$).

$*$	4	0	0	-18	10
$*$	$*$	0	5	0	3
$*$	$*$	$*$	0	0	1
$*$	$*$	$*$	$*$	12	1
$*$	$*$	$*$	$*$	$*$	-11
$*$	$*$	$*$	$*$	$*$	$*$

Looking at the table, the orthogonal pairs are:

(i,iii), (i, iv), (ii,iii), (ii,v), (iii,iv), (iii,v).

2. (5 points) Find a unit vector which is orthogonal to both of the vectors

$$(1,1,1) \text{ and } (1,0,1).$$

Sol'n

We know a vector orthogonal to both of these vectors is given by the cross product of the two given vectors:

$$(\widehat{i}+\widehat{j}+\widehat{k})\times(\widehat{i}+\widehat{k}) = (\widehat{i}+\widehat{k})\times(\widehat{i}+\widehat{k})+\widehat{j}\times(\widehat{i}+\widehat{k}) = 0+\widehat{j}\times(\widehat{i}+\widehat{k}) = \widehat{j}\times\widehat{i}+\widehat{j}\times\widehat{k} = -\widehat{k}+\widehat{i}.$$

The only problem with this vector is that it is not a unit vector (its length is not equal to 1), therefore we normalize it:

$$\boxed{\frac{\widehat{i}-\widehat{k}}{\sqrt{2}}.}$$

3. Let $c(t)$ be the curve $(2t, \sqrt{3}t^2, t^3)$ be a curve.

   (a) (4 points) What is the velocity of $c(t)$ at time $t = 2$?

Sol'n

$$c'(t) = (2, 2\sqrt{3}t, 3t^2) \Rightarrow c'(2) = (2, 2\sqrt{3} \cdot 2, 3 \cdot 2^2) = \boxed{(2, 4\sqrt{3}, 12).}$$

   (b) (3 points) What is the speed of $c(t)$ at time $t = 2$?

Sol'n

In general, the speed at a point is simply the length of the velocity of that point. Therefore, our answer is:

$$\|c'(2)\| = \sqrt{4 + 3 \cdot 16 + 144} = \sqrt{52 + 144} = \sqrt{196} = \boxed{14}.$$

   (c) (3 points) Compute of the length of this curve between $t = 1$ and $t = 3$.

Sol'n

The length of a curve defined over an interval is the integral of the speed of the curve over that same interval. The speed of the curve $c(t)$ is

$$\|c'(t)\| = \|(2, 2\sqrt{3}t, 3t^2)\| = \sqrt{4 + 12t^2 + 9t^4} = \sqrt{(2 + 3t^2)^2} = 2 + 3t^2.$$

Therefore, the length is

$$\int_1^3 (2 + 3t^2)\text{dt} = (2t + t^3)\Big|_1^3 = (2 \cdot 3 + 3^3) - (2 \cdot 1 + 1^3) = (6 + 27) - (3) = 3 + 27 = \boxed{30}.$$

4. (5 points) Give an example of a function

$$f : \mathbb{R}^2 \to \mathbb{R}^3.$$

Sol'n

One possible answer is $\boxed{f(x, y) = (x, y, xy)}$.

5. (10 points) Show that the following limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

Sol'n

Let $f(x,y)$ be the function in question. Then

$$\lim_{t\to 0)} f(t,0) = \lim_{t\to 0)} \frac{t^2-0^2}{t^2+0} = \lim_{t\to 0)} \frac{t^2}{t^2} = \lim_{t\to 0)} 1 = 1.$$

However, meanwhile

$$\lim_{t\to 0)} f(0,t) = \lim_{t\to 0)} \frac{0^2-t^2}{0+t^2} = \lim_{t\to 0)} \frac{-t^2}{t^2} = \lim_{t\to 0)} (-1) = -1.$$

Therefore, no limit can exist.

6. (10 points) Verify that the chain rule holds for the functions

$$F(x,y,z) = (xz, yz) \text{ and } G(u,v) = (u^2, uv, v^2).$$

(for the composition $F \circ G$).

Sol'n

The chain rule states that

$$dF_{G(u,v)} \cdot dG_{(u,v)} = d(F \circ G)_{u,v}. \tag{1}$$

The Jacobian of F is

$$dF_{(x,y,z)} = \begin{pmatrix} z & 0 & x \\ 0 & z & y \end{pmatrix}$$

and the Jacobian of G is

$$dG_{(u,v)} = \begin{pmatrix} 2u & 0 \\ v & u \\ 0 & 2v \end{pmatrix}.$$

Meanwhile

$$dF_{G(u,v)} = dF_{(u^2,uv,v^2)} = \begin{pmatrix} v^2 & 0 & u^2 \\ 0 & v^2 & uv \end{pmatrix}.$$

Taking their product gives

$$dF_{G(u,v)} \cdot dG_{(u,v)} = \begin{pmatrix} v^2 & 0 & u^2 \\ 0 & v^2 & uv \end{pmatrix} \cdot \begin{pmatrix} 2u & 0 \\ v & u \\ 0 & 2v \end{pmatrix} = \begin{pmatrix} 2uv^2 & 2u^2v \\ v^3 & 3uv^2 \end{pmatrix}. \tag{2}$$

To verify the chain rule, we still need to compute the right hand side of equation (1). First note that

$$(F \circ G)(u,v) = F(G(u,v)) = F(u^2, uv, v^2) = (u^2 \cdot v^2, uv \cdot v^2) = (u^2v^2, uv^3).$$

Therefore

$$d(F \circ G)_{(u,v)} = \begin{pmatrix} 2uv^2 & 2u^2v \\ v^3 & 3uv^2 \end{pmatrix}. \tag{3}$$

Since equation (2) matches equation (3) the chain rule has been verified.

7. (10 points) Calculate the equation of the tangent plane to the graph of the equation

$$x^4z + y^5 + y^3z^2 = 3$$

at the point $(1, 1, 1)$.

Sol'n

Let $f(x, y, z) = x^4z + y^5 + y^3z^2$, be the function in question. Then

$$\nabla f_{(1,1,1)} = (4x^3z, 5y^4 + 3y^2z^2, x^4 + 2y^3z)\,|_{(1,1,1)} = (4, 5 + 3, 1 + 2) = (4, 8, 3).$$

Therefore the equation of the tangent plane is given by

$$\nabla f_{(1,1,1)} \bullet (x - 1, y - 1, z - 1) = 4(x - 1) + 8(y - 1) + 3(z - 1) = 0.$$

This answer would have been fine, but if you prefered to bring the scalar to the other side you would have gotten

$$\boxed{4x + 8y + 3z = 15.}$$

8. (20 points) Find and classify all critical points of the function

$$f(x, y) = x^3 + y^3 - 12xy.$$

Sol'n

The gradient of f is $(3x^2 - 12y, 3y^2 - 12x)$. This equals zero if and only if $x^2 = 4y$ and $y^2 = 4x$. Substituting the first equation into the second we see that $x^4/16 = 4x$ which holds if and only if $x = 0$ or $x = 4$. Going backwards, if $x = 0$, then $y = 0$ as well. On the other hand if $x = 4$, then $16 = 4y$ implies that $y = 4$ as well. Therefore we have two critical points

$$\boxed{(0,0) \qquad \text{and} \qquad (4,4).}$$

The hessian of f is

$$\begin{pmatrix} 6x & -12 \\ -12 & 6y \end{pmatrix}.$$

When evaluating this at the first critical point, $(0, 0)$ we are left with the matrix

$$\begin{pmatrix} 0 & -12 \\ -12 & 0 \end{pmatrix}.$$

which has negative determinant and as such $\boxed{(0, 0) \text{ is a saddle point}}$. Now evaluating the Hessian at the other critical point, $(4, 4)$ yields the matrix

$$\begin{pmatrix} 24 & -12 \\ -12 & 24 \end{pmatrix}$$

which clearly has positive determinant (since $24 > 12$). Since the upper left hand entry, 24, is also positive the critical point $\boxed{(4, 4) \text{ is a local min}}$.

9. (20 points) Some evil dude lights the earth on fire. We put a coordinate system on the universe so that the earth is given by

$$x^2 + y^2 + z^2 = 1.$$

Mark, being cool and all, can safely measure the temperature at a point (x, y, z) on earth to be given by

$$x + 2y + z. \quad [1]$$

At what point should he go to take out the bad guy? (Of course the bad guy is where the earth is hottest.) Meanwhile, some wimpy non-math major, filled with fear, decides to run to where it's coolest. What point should they run to?

Sol'n

This is a LaGrangian multiplier problem with working equation $f(x, y, z) = x + 2y + z$ and constraint $x^2 + y^2 + z^2 = 1$. To use the LaGrangian multiplier method we set the gradient of the working equation to be proportional to the gradient of the constraint:

$$(2x, 2y, 2z) = \nabla(x^2 + y^2 + z^2) = \lambda\nabla(x + 2y + z) = \lambda(1, 2, 1). \tag{4}$$

Now equating components in equation 4 gives us three equations:

$$2x = \lambda \tag{5}$$
$$2y = 2\lambda \tag{6}$$
$$2z = \lambda \tag{7}$$

and coupling with the constraint gives us a fourth equation to play with:

$$x^2 + y^2 + z^2 = 1. \tag{8}$$

Glancing at the first three equations tells us that

$$x = z \tag{9}$$

(since they are both equal to $\lambda/2$) and

$$y = 2x. \tag{10}$$

Now plugging our new found facts, (9) and (10), into the constraint, (8), gives us

$$x^2 + y^2 + z^2 = x^2 + (2x)^2 + x^2 = 1 \Rightarrow x^2 + 4x^2 + x^2 = 6x^2 = 1.$$

Hence

$$x = \pm\frac{1}{\sqrt{6}}.$$

Now back substituting this into equation (9) tells us that

$$z = \pm\frac{1}{\sqrt{6}}$$

as well and back substituting into equation (10) tells us that

$$y = 2x = \pm\frac{2}{\sqrt{6}}.$$

[1] He can do this by assuming the function to be linear and simply doing a few readings in his lab!

Therefore we have found two critical points:

$$\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \qquad \text{and} \qquad \left(-\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right).$$

Clearly the working equation is positive on the first critical point and negative on the second. Therefore we have that

$$\boxed{\text{the hottest point is } \left(\tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\right) \text{ and the coldest is } \left(-\tfrac{1}{\sqrt{6}}, -\tfrac{2}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right).}$$

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10. (5 points) **Bonus!** Calculate the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{pmatrix}.$$

Sol'n

There were several ways of doing this (like every determinant!). Every way one proceeds you can see that the answer is -1.

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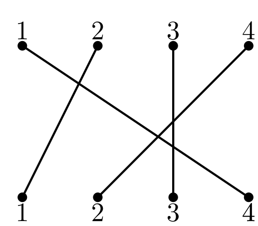
11. (5 points) **Bonus!** Calculate the sign of the permutation

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{matrix}.$$

Sol'n

The diagram for this permutation is



which has 4 crossings. Since the number 4 is even the permutation is even and has sign $+1$.
