Answers to Exam from Fall '07

Mark O'Brien

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1. (a) We parameterize the unit circle, C, in the usual way, $c(t) = (\cos(t), \sin(t)), 0 \leq t \leq 2\pi$. Then

$$\begin{aligned}\mathbf{F}(c(t)) &= \left\langle \frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{-\cos(t)}{\cos^2(t) + \sin^2(t)} \right\rangle \\ &= \left\langle \frac{-\sin(t)}{1}, \frac{-\cos(t)}{1} \right\rangle \\ &= \langle -\sin(t), -\cos(t) \rangle\end{aligned}$$

and

$$c'(t) = \langle -\sin(t), \cos(t) \rangle$$

So that

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F} \cdot c'(t) dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt \\ &= \int_0^{2\pi} dt \\ &= 2\pi.\end{aligned}$$

   (b) The vector fields is not conservative since part (a) shows that not every closed path integral of this vector field is 0.

2. (a)

$$\vec{r}\,'(t) = \langle 1, e^t, 2t \rangle$$

   (b)

$$\|\vec{r}\,'(t)\| = \sqrt{1 + e^{2t} + 4t^2}$$

(c)

$$\vec{r}''(t) = \langle 0, e^t, 2\rangle$$

(d)

$$\int_0^3 \|\vec{r}'(t)\| dt = \int_0^3 \sqrt{1 + e^{2t} + 4t^2} dt$$

3. Let $f(x, y, z) = ze^{x^2-y^2}$. Then

$$\begin{aligned}\nabla(f) &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \\ &= \left\langle 2xze^{x^2-y^2}, -2yze^{x^2-y^2}, e^{x^2-y^2} \right\rangle\end{aligned}$$

So that $\nabla(f)(1, -1, 2) = \langle 4, 4, 1\rangle$ and the equation of the tangent plain is given by

$$\nabla(f)(1, -1, 2) \cdot \langle x - 1, y + 1, z - 2\rangle = 4(x - 1) + 4(y + 1) + (z - 2) = 0$$

4. First we must find the critical points of f. To do this we find the points where the gradient of f is 0. $\nabla(f) = \langle y - x, x + y^2 - 2\rangle$. The first component implies that the x coordinate of a critical point is equal to its y component. Using this in the second component of $\nabla(f)$ gives us $x + x^2 - 2 = 0$ i.e. $x = -2$ or $x = 1$. Thus the coordinates of the two critical points are $(1, 1), (-2, -2)$.

   Now to figure out the nature of these critical points we need to use the second derivative test. The Hessian is

$$\begin{bmatrix} -1 & 1 \\ 1 & 2y \end{bmatrix}$$

   which has determinant $-2y - 1$ which is negative when evaluated at the first critical point $(1, 1)$ and positive when evaluate at the second critical point $(-2, -2)$. Thus the critical point $(1, 1)$ is a saddle point while the critical point $(-2, -2)$ is either a local min or a local max. To figure out which we need only need to note that $f_{x,x}(-2, -2)$ is negative and as such $(-2, -2)$ is a local maximum.

5.

$$\begin{aligned}\int\int_{\mathcal{D}} f(x,y)dxdy &= \int_0^1 \int_0^{y^2} e^{y^3} dxdy \\ &= \int_0^1 y^2 e^{y^3} dy \\ &= \frac{1}{3}\int_0^1 e^u du \\ &= \frac{1}{3}(e-1)\end{aligned}$$

6. (a)

$$\begin{aligned}\overrightarrow{DA} &= \langle 1,-1,0\rangle \\ \overrightarrow{DC} &= \langle 1,2,0\rangle\end{aligned}$$

and

$$\overrightarrow{DA} \times \overrightarrow{DC} = 3k.$$

Thus the parallelogram has area 3.

(b) We are trying to find the closed path integral of the vector field given by $\mathbf{F} = \langle -y, x\rangle$ over the given path. By Green's theorem this is the same as finding double integral of the curl of $\mathbf{F}$'s k component, which is $\frac{\partial(x)}{\partial x} - \frac{\partial(y)}{\partial y} = 2$, integrated over the parallelogram, P from the first part of this problem.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{s} =& \int\int_P 2dxdy \\ =&2\int\int_P dxdy \\ =&2\,\text{Area}(P) \\ =&2\cdot 3 \\ =&6.\end{aligned}$$

(In this problem it is possible to do the given line integral explicitly in four steps, maybe you should try it to get a stronger sense of how much easier using Green's theorem was in this problem).

7. We need to calculate a triple integral in this problem. Since a cylinder shows up explicitly in the statement of the problem, cylindrical coordinates would seem like the obvious choice. The bounds of the domain $\mathcal{B}$ in cylindrical coordinates are given by

$$\begin{aligned}&0 \le z \le r^2 \\ &0 \le r \le 1 \\ &0 \le \theta \le 2\pi.\end{aligned}$$

This is because the region is trapped between the cartesian equations $z = 0$ and $z = x^2 + y^2$ and projects down to the unit disk in the x, y plane. Moreover the function we are integrating, x^2 is equal to $r^2 \cos^2(\theta)$ in cylindrical coordinates.

Now is the is the simple matter of integrating.

$$\begin{aligned}\iiint_{\mathcal{B}} x^2 dV = &\iiint_{\mathcal{B}} \left(r^2 \cos^2(\theta)\right) (r dr d\theta dz)\\ =&\int_0^{2\pi}\int_0^1\int_0^{r^2} r^3 \cos^2(\theta) dz dr d\theta\\ =&\int_0^{2\pi}\int_0^1 r^5 \cos^2(\theta) dz dr d\theta\\ =&\frac{1}{6}\int_0^{2\pi} \cos^2(\theta) dz dr d\theta\\ =&\frac{\pi}{6}.\end{aligned}$$

8. (a)

$$\begin{aligned} curl\,(\mathbf{F}) =& \det\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & yz & z^3 \end{bmatrix}\\ =& -y\mathbf{i} - 0\mathbf{j} + x^2\mathbf{k}\\ =& -y\mathbf{i} + x^2\mathbf{k}\end{aligned}$$

(b) No, this vector field can not be conservative since the curl of the vector field does not equal the zero vector.

(c) $\text{div}(\mathbf{F}) = 2xy + z + 3z^2$

(d) The divergence of a vector field that is the curl of another vector field is always the zero function (this can be easily checked in this special case).

9. We can solve the equation determining the surface explicitly for z as $z = 5 + 2y - 2x$. Thus we can parameterize the surface as $\Phi(u, v) = (u, v, 5 + 2v - 2u)$ with $0 < u, v < 1$.

We first calculate the normal vector

$$\begin{aligned}\Phi_u = \langle 1, 0, -2\rangle \quad , \quad \Phi_v = \langle 0, 1, 2\rangle&\\ \Rightarrow \Phi_u \times \Phi_v =&(\mathbf{i} - 2\mathbf{k}) \times (\mathbf{j} + 2\mathbf{k})\\ =&\mathbf{i} \times \mathbf{j} + 2\mathbf{i} \times \mathbf{k} - 2\mathbf{k} \times \mathbf{j} - 4\mathbf{k} \times \mathbf{k}\\ =&\mathbf{k} - 2\mathbf{j} + 2\mathbf{i} - 4\overrightarrow{0}.\\ =&\langle 2, -2, 1\rangle\end{aligned}$$

(notice this is indeed the upward pointing normal vector, if it wasn't we would have to replace this vector by its negative). Next we note that $\mathbf{F}(u,v) = \langle u, v, 5+2v-2u\rangle$ and

$$\begin{aligned}\mathbf{F}(u,v)\cdot(\Phi_u\times\Phi_v) &= \langle u, v, 5+2v-2u\rangle\cdot\langle 2,-2,1\rangle\\ &= 2u-2v+5+2v-2u\\ &= 5.\end{aligned}$$

Finally we integrate

$$\begin{aligned}\int\!\!\int_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^1\int_0^1 \mathbf{F}(u,v)\cdot(\Phi_u\times\Phi_v)\ dudv\\ &= \int_0^1\int_0^1 5\,dudv\\ &= 5\end{aligned}$$